Stochastic lot-sizing problem: a joint chance-constrained programming approach

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Plan

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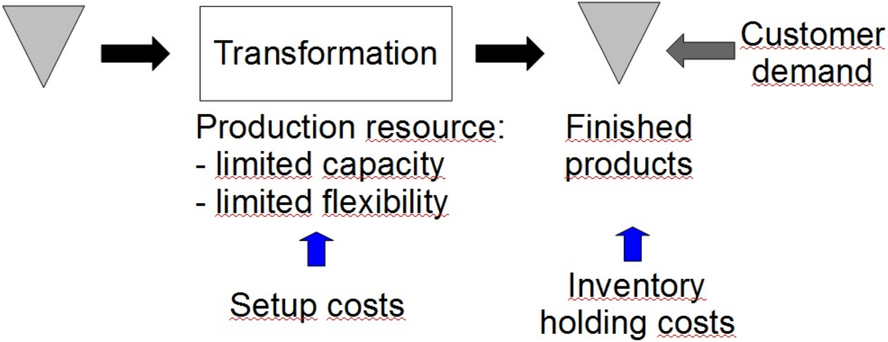
Production system



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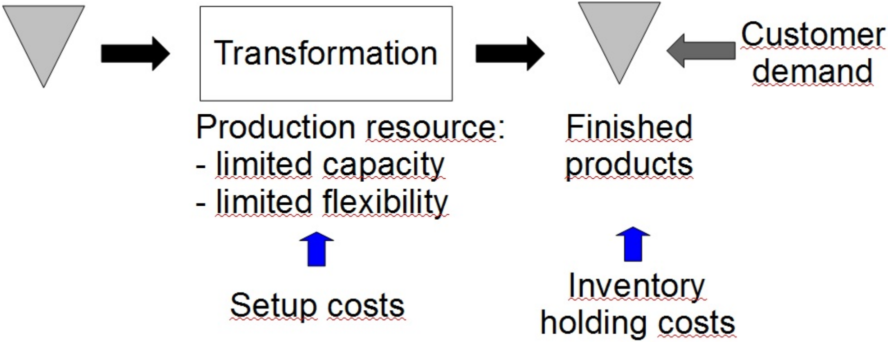

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Production planning
Decide when and how much to produce on the production resource

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- minimize setup costs
  $\rightarrow$ A single production lot of large size
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General lot sizing problem

Plan production so as to:

- satisfy customer demand
- minimize setup and inventory holding costs

Small illustrative example

- Planning horizon: $T = 10$ days
- One product with demand:

Day	1	2	3	4	5	6	7	8	9	10
Demand	0	10	0	0	0	10	0	0	10	20

- Production resource
  Capacity: 50 units par period
- Costs
  - Setup costs: 500€
  - Inventory holding costs: 5€ per unit per period

Small illustrative example

Objective 1: minimize setup costs

0	1	2	3	4	5	6	7	8	9	10
0	0	50	0	0	0	0	0	0	0	0

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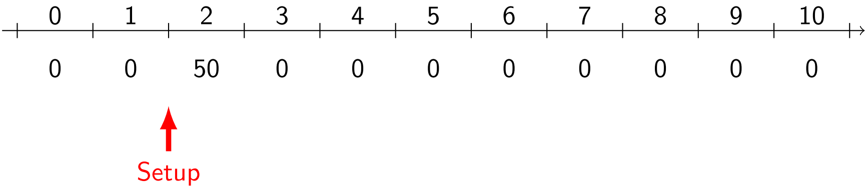

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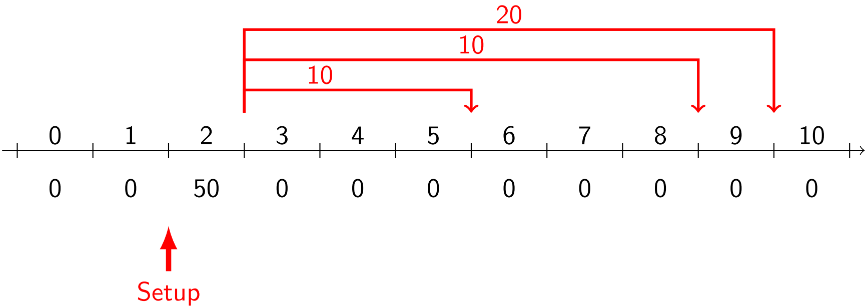

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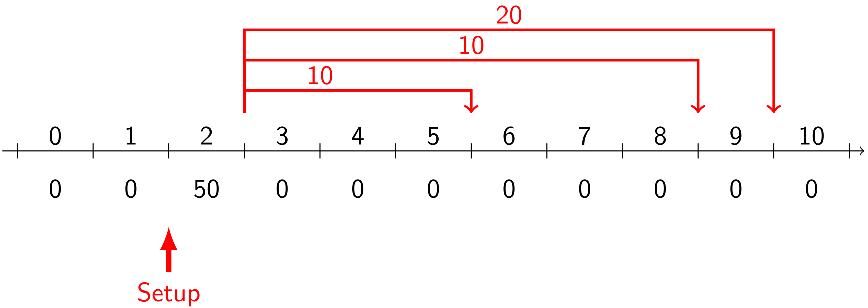

Setup: 500€
Inventory: 1350€
→ Total cost= 1850€

Small illustrative example

Objective 2: minimize inventory holding costs

0	1	2	3	4	5	6	7	8	9	10
0	0	10	0	0	0	10	0	0	10	20

Small illustrative example

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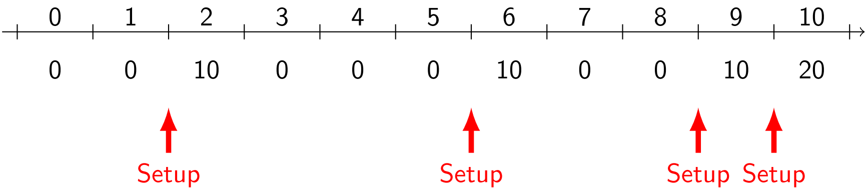

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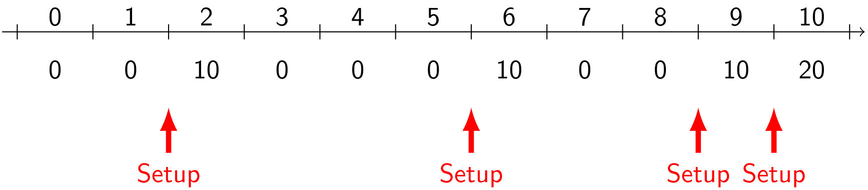


Setup: 2000€

Inventory: 0€

$\rightarrow$ Total cost= 2000€

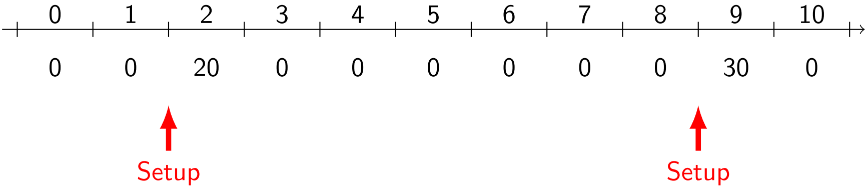
Small illustrative example

Objective 3: minimize setup and inventory holding costs

0	1	2	3	4	5	6	7	8	9	10
0	0	20	0	0	0	0	0	0	30	0

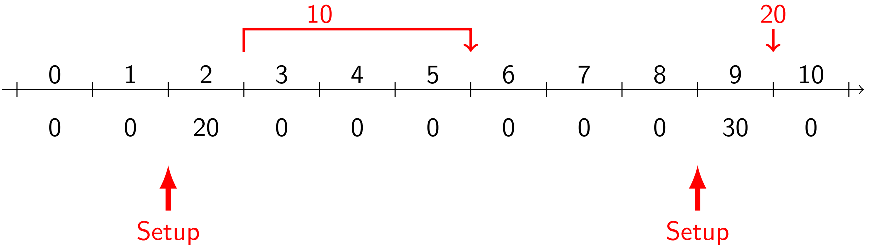
Small illustrative example

Objective 3: minimize setup and inventory holding costs



Small illustrative example

Objective 3: minimize setup and inventory holding costs



Setup: 1000€
Inventory: 250€
$\rightarrow$ Total cost= 1250 €

Deterministic single-item capacitated lot-sizing problem

Parameters

- s, h: setup / inventory holding cost
- c: production capacity
- d_t: demand in period $t = 1...T$
  dc_t: cumulated demand over periods $1...t$

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- x_t: production variable
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MILP Formulation

$$\begin{cases} Z_{DET} = min \sum_{t=1}^{T} sy_t + \sum_{t=1}^{T} h(\sum_{\tau=1}^{t} x_\tau - dc_t) & \\ \quad x_t \leq cy_t & \forall t \\ \quad \sum_{\tau=1}^{t} x_\tau - dc_t \geq 0 & \forall t \\ \quad x_t \geq 0, y_t \in \{0,1\} & \forall t \end{cases}$$

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Stochastic single-item CLSP

Stochastic demand

- Demand not perfectly known in advance due e.g. to forecasting errors
- Deterministic value dc_t replaced by random variable DC_t
- Assumption: known probability distribution for DC_t

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Main difficulty

Possible violation of the demand satisfaction constraints

Modeling alternatives: stockout risk management

How to manage the violation of the demand satisfaction constraints ?

1. Allow violation and penalize it
   - Allow backlog
   - Estimate backlog penalty
   - Minimize expected backlogging costs

[Vargas, 2009], [Piperagkas *et al*, 2012]

[Tempelmeier, 2007], [Guan, 2011]

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2. Limit the probability of violation
   - Define a minimum acceptable service level
   - Introduce chance constraints

[Chen, 2007], [Kuçukyavuz, 2012]

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Individual chance constraints

Stochastic formulation with individual chance constraints

$\sim$ Impose a minimum value p to the service level within each period

$$\begin{cases} Z_{ST1} = min \sum_{t=1}^{T} sy_t + \sum_{t=1}^{T} h(\sum_{\tau=1}^{t} x_\tau - \mathbb{E}\left[DC_t\right]) & \\ \qquad x_t \leq cy_t & \forall t \\ \qquad \Pr\left(\sum_{\tau=1}^{t} x_\tau - DC_t \geq 0\right) \geq p & \forall t \\ \qquad x_t \geq 0, y_t \in \{0,1\} & \forall t \end{cases}$$

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Deterministic equivalent

$$\begin{cases} Z_{ST1} = min \sum_{t=1}^{T} sy_t + \sum_{t=1}^{T} h(\sum_{\tau=1}^{t} x_\tau - \mathbb{E}[DC_t]) & \\ \quad x_t \leq cy_t & \forall t \\ \quad \sum_{\tau=1}^{t} x_\tau \geq F_{DC_t}^{-1}(p) & \forall t \\ \quad x_t \geq 0, y_t \in \{0,1\} & \forall t \end{cases}$$

Joint chance constraints

Stochastic formulation with joint chance constraints

$\sim$ Impose a minimum value p to the service level for the planning horizon

$$\begin{cases} Z_{ST2} = min \sum_{t=1}^{T} sy_t + \sum_{t=1}^{T} h(\sum_{\tau=1}^{t} x_\tau - \mathbb{E}\left[DC_t\right)\right] & \\ \quad x_t \leq cy_t & \forall t \\ \quad \Pr\left(\sum_{\tau=1}^{t} x_\tau - DC_t \geq 0 \quad \forall t\right) \geq p & \\ \quad x_t \geq 0, y_t \in \{0,1\} & \forall t \end{cases}$$

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A challenging problem

- Computational difficulty to check feasibility of a given solution
- Non convexity of the solution space of the continuous relaxation

[Nemirovski and Shapiro, 2006], [Luedtke and Ahmed, 2008]

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Sample approximation approach

- Discretize the probability distributions
- Solve an LP/MILP approximation
- No guarantee of finding a feasible solution

[Luedtke and Ahmed, 2008], [Kuçukyavuz, 2012]

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Sample approximation for lot-sizing

Monte Carlo sample of the random demand vector DC

- N sampled scenarios: $DC^1, ..., DC^i, ..., DC^N$
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Approximation of the joint probability

Given a production plan x:

- for each scenario i: check whether all demand satisfaction constraints are satisfied
- count the total number N_{sat} of such scenarios
- estimate the probability by N_{sat}/N

$$\Pr\left(\sum_{\tau=1}^{t} x_\tau - DC_t \geq 0 \quad \forall t\right) \approx \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}\left(\sum_{\tau=1}^{t} x_\tau - DC_t^i \geq 0 \quad \forall t\right)$$

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New binary decision variables

$$\alpha_i = \begin{cases} 1 & \text{if all demand satisfaction constraints are satisfied in scenario } i \\ 0 & \text{otherwise} \end{cases}$$

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MILP formulation

$$\begin{cases} Z_{SA} = min \sum_{t=1}^{T} sy_t + \sum_{t=1}^{T} h(\sum_{\tau=1}^{t} x_\tau - \mathbb{E}\left[DC_t\right]) & \\ \quad x_t \leq cy_t & \forall t \\ \quad \sum_{\tau=1}^{t} x_\tau \geq DC_t^i \alpha_i & \forall t, \forall i \\ \quad \frac{1}{N} \sum_{i=1}^{N} \alpha_i \geq p & \\ \quad x_t \geq 0, y_t \in \{0,1\} & \forall t \\ \quad \alpha_i \in \{0,1\} & \forall i \end{cases}$$

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Theoretical results

- No guarantee to find a feasible solution of the original problem
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$\rightarrow$ Long computation times

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Our proposal

- Assume D_1 is independent of $D_2, ..., D_T$
- Avoid the use of variables α_i through a partial sample approximation approach

$\rightarrow$ Shorter computation times

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Assumption

D_1: random variable independent of $D_2, ..., D_T$

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Monte Carlo sampling of the random vector ΔC

- ΔC: cumulated demand over periods 2...T
  $\Delta C = (0, D_2, D_2 + D_3, ..., \sum_{t=2}^{T} D_t)$
  NB: $DC_t = \Delta C_t + D_1$

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- N equiprobable sampled scenarios: $\Delta C^1, ..., \Delta C^i, ..., \Delta C^N$

Partial sample approximation for lot-sizing

Approximation of the joint probability

Given a production plan x:

- for each scenario i: compute the probability that all demand satisfaction constraints are satisfied
  $\pi_i = \Pr\left(\sum_{\tau=1}^{t} x_\tau - \Delta C_t^i \geq D_1 \quad \forall t\right)$
- estimate the joint probability as the expected value of π_i over all scenarios

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MIP formulation

$$\begin{cases} Z_{PSA} = min \sum_{t=1}^{T} sy_t + \sum_{t=1}^{T} h(\sum_{\tau=1}^{t} x_\tau - \mathbb{E}[DC_t]) & \\ x_t \leq cy_t & \forall t \\ \pi_i = \Pr\left(\sum_{\tau=1}^{t} x_\tau - \Delta C_t^i \geq D_1 \quad \forall t\right) & \forall i \\ \frac{1}{N}\sum_{i=1}^{N} \pi_i \geq p & \\ x_t \geq 0, y_t \in \{0,1\} & \forall t \\ \pi_i \leq 1 & \forall i \end{cases}$$

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Computational experiments

Objective

Compare the partial sample approximation with the sample approximation

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90 small instances

- $T = 10$ periods
- Costs: $h = 1$, $S = 50$, Capacity: $c = 100$
- Demand $D_1, ..., D_T$: independent variables following a U(10,50) distribution
- Service level p: 0.85, 0.90, 0.95
- Sample size N: 100, 1000, 10000

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Settings

- MILP solver: CPLEX 12.6
- Computer: Intel Core i5(2.6GHz), 4Go of RAM, Windows 7

Computational experiments

Post-optimization analysis

Check feasibility of the production plan with respect to the joint probabilistic constraint

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Check feasibility of the production plan with respect to the joint probabilistic constraint

1. Estimate $Prob = \Pr\left(\sum_{\tau=1}^{t} x_{\tau}^{*} - DC_t \geq 0 \quad \forall t\right)$
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   - Simulation over 100000 sampled scenarios
   - For each scenario: check whether all demand satisfaction constraints are satisfied
   - $Prob$= proportion of scenarios without any violation
2. if $Prob \geq p$: feasible production plan

Preliminary results

p	N	SA Prob	SA Cost	SA Time	PSA Prob	PSA Cost	PSA Time
0.85	100	0.762	769	0.6s	0.836	825	0.4s
	1000	0.841	821	1101.5s	0.857	840	5.8s
	10000	_	_	_	0.856	837	196.1s
0.90	100	0.817	813	0.4s	0.883	878	0.4s
	1000	0.888	878	172s	0.905	896	4.6s
	10000	_	_	_	0.903	893	120.4s
0.95	100	0.882	887	0.4s	0.932	954	0.3s
	1000	0.935	954	9.8s	0.950	976	1.8s
	10000	_	_	_	0.950	974	60.1s

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Conclusion and perspectives

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- Joint chance-constrained programming formulation
- New solution approach: partial sample approximation
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Perspectives

- Confirm results on a larger set of instances
- Extend to normally distributed demands

Thank you for your attention !