Synthèse sur les problèmes de lot sizing et perspectives

28ème Journée Francilienne de Recherche Opérationnelle

Stéphane DAUZERE-PERES
(avec l'aide de Nabil ABSI et Nadjib BRAHIMI)
Département Sciences de la Fabrication et Logistique
Centre Microélectronique de Provence – Site Georges Charpak
Ecole des Mines de Saint-Etienne
Dauzere-Peres@emse.fr

Outline

- General introduction
- Problem modeling
- Classifications
- Disaggregate and shortest path formulations
- Solving the uncapacitated single-item lot-sizing problem
- Solving the multi-item lot-sizing problem
- Some extensions of lot-sizing problems
- Modeling and solving multi-level lot-sizing problems
- Integrating lot-sizing decisions with other decisions

General Introduction

Lot Sizing = Determination of sizes of (production or distribution) lots, i.e. ***quantities* of products** (to produce or distribute).

This talk is concerned with "**Dynamic**" lot sizing, i.e. **demands d_t** are varying on a time horizon decomposed in T periods.

The goal is to determine a plan, i.e. the **quantity X_t** at each period t, which results in an **inventory level I_t** following the well-known inventory balance equation:

$$I_t = I_{t-1} + X_t - d_t$$

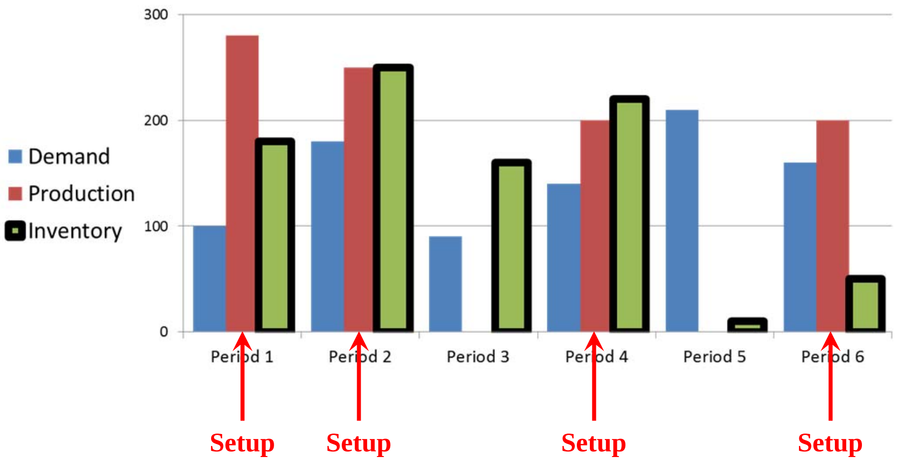
Lot Sizing usually means setups,

i.e. **setup variable Y_t** = 1 if $X_t > 0$ and, $Y_t = 0$ otherwise.

Basic trade-off: *inventory cost vs setup cost*, however production cost vs inventory cost vs setup cost is also possible.

General Introduction

Example of a plan on a single item.



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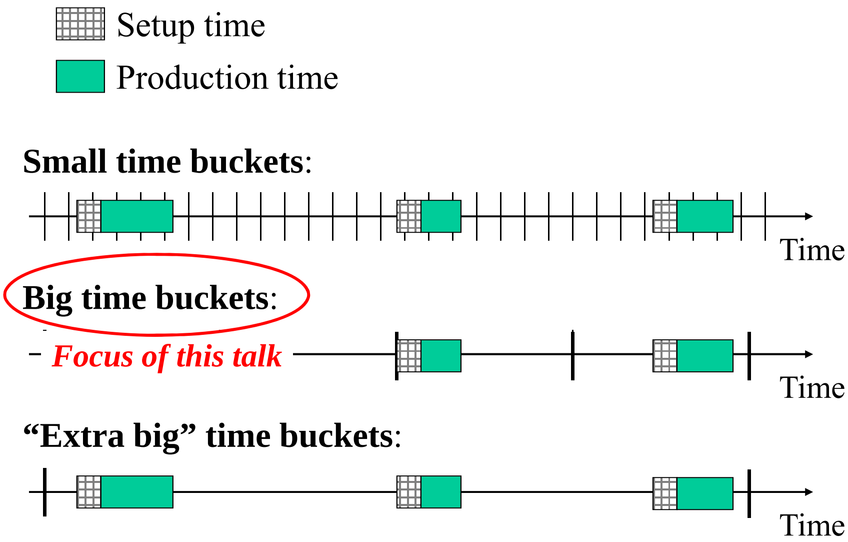
General Introduction
Modeling Setups



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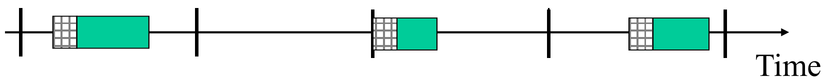
General Introduction
Modeling Setups

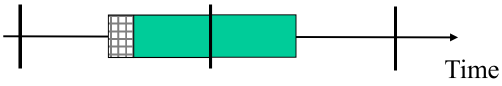
- In big time bucket models, there is one setup for each period in which the product is produced.
- **Classical big time bucket models:**
  - The **Uncapacitated** Lot-Sizing Problem (ULSP) (Wagner-Whitin problem)
  - The **Capacitated** Lot-Sizing Problem (CLSP)

Big time buckets:



General Introduction
Modeling Setups

- In multi-item production, a setup can sometimes be saved by letting the same product produced last in a given period to be produced first in the next period (production **carryovers**):

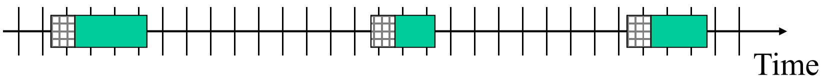


- Most big time bucket models do not take into account carryovers and, in such cases, small time bucket models may be better suited.
- However, Aras and Swanson [1982] proposed a modification so that production carryovers can also be handled by big time bucket models.

General Introduction
Modeling Setups

- When time buckets are small, there is only one setup even if a production lot lasts for several periods
  → Means of considering scheduling decisions in lot-sizing models
- **Some small time bucket models:**
  - The Discrete Lot-sizing and Scheduling Problem (DLSP),
    → *See presentation of C. Gicquel,*
  - The Continuous Setup Lot-sizing Problem (CSLP),
  - The Proportional Lot-sizing and Scheduling Problem (PLSP),
  - The General Lot-sizing and Scheduling Problem (GLSP).

Small time buckets:



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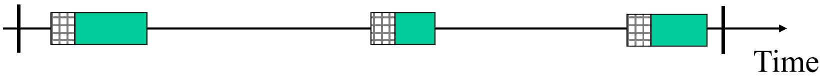
General Introduction

Modeling Setups

- Time buckets are so big that each individual setup is not important for the total production capacity; there will normally be several setups for each product in the period.
- Setups are therefore not considered at all ("**aggregate production planning**"), and a pure LP model can be used.
- "Average setup time" is either deducted from the total capacity, or production time per unit is increased in order to compensate for the setup time.

"Extra big" time buckets:



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General Introduction

Lot-Sizing problems are useful in production and distribution, where many practical problems can be found.

The study of lot-sizing problems has been driven by the development of APS (Advanced Planning Systems).

General Introduction

Renewal from the mid-2000's, illustrated by the success of the International Workshop on Lot Sizing (IWLS) started in Gardanne in 2010.

IWLS 2010 International Workshop on Lot Sizing

August 25th - 27th, 2010 Gardanne, France

General Introduction

IWLS 2011 International Workshop on Lot Sizing

August 24th - 26th, 2011 Istanbul, Turkey

IWLS 2012 International Workshop on Lot Sizing

August 27th - 29th, 2012 Rotterdam, The Netherlands

IWLS 2013 International Workshop on Lot Sizing

August 26th - 28th, 2013 Brussels, Belgium

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Problem modeling

Uncapacitated Lot-Sizing Problem (USLP)

- **Decision variables:**
  - Y_t = 1 if production (setup) in period t, = 0 otherwise
  - X_t = Production quantity in period t
  - I_t = Inventory at the end of period t
- **Parameters:**
  - d_t = Demand in period t
  - v_t = Variable production cost per unit in period t
  - s_t = Setup cost in period t
  - c_t = Inventory holding cost per unit on stock at the end of period t

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Problem modeling

Uncapacitated Lot-Sizing Problem (USLP)

Min $\quad \sum_{t=1}^{T}\left(s_t Y_t + v_t X_t + c_t I_t\right)$

subject to :

$$
\begin{aligned}
& X_t + I_{t-1} - I_t = d_t && \forall t \\
& X_t \le M_t Y_t && \forall t \\
& X_t, I_t \ge 0 && \forall t \\
& Y_t = \{0,1\} && \forall t
\end{aligned}
$$

Where M_t is an upper bound of X_t, e.g. $M_t = d_t + \ldots + d_T$.

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Problem modeling
Uncapacitated Lot-Sizing Problem (USLP)

Analysis of the Linear Relaxation, i.e. when $Y \geq 0$.

What happens to Y_t in an optimal solution?

→ Y_t will be equal to X_t / M_t (as long as setup cost $s_t > 0$),

→ If M_t is much larger than X_t, then Y_t will be close to zero,

→ Optimal value of LP relaxation will be much lower than optimal value of original problem.

→ Efficiency of standard solvers depend on choice of M_t.

→ Various valid inequalities have been proposed.

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Problem modeling
Uncapacitated Lot-Sizing Problem (USLP)

Valid inequality *(l, S)*. (Barany, Van Roy and Wolsey, 1984)
By definition, if there is a production in period t (i.e. $X_t > 0$ and $Y_t = 1$), then d_t is produced in t.

→ The following inequality is valid: $X_t \leq d_t Y_t + I_t$.

This inequality can be generalized to two periods:

$$X_t + X_{t+1} \leq (d_t + d_{t+1})Y_t + d_{t+1} Y_{t+1} + I_{t+1}$$

and to any number of periods.

***(l, S)* inequalities (exponential number) can be used to fully describe the convex hull of the USLP.**

$$\sum_{l \in S} X_l \leq \sum_{l \in S} (\sum_{k=l}^{t} d_k) Y_l + I_t \qquad \forall t, \forall S \subseteq \{1, \ldots, t\}$$

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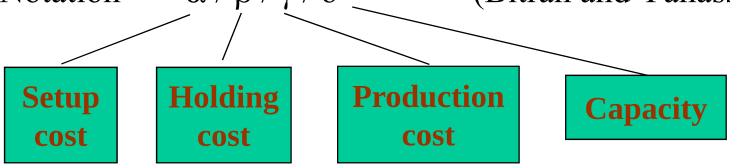
Problem modeling
Single-item Lot-Sizing Problems

Uncapacitated can be solved in $O(T^2)$ (Wagner and Whitin, 1958) and $O(T \log T)$ (Federgruen and Tzur, 1991) (Wagelmans, van Hoesel and Kolen, 1992) (Aggarwal and Park, 1993).

Capacited single-item lot sizing:

Notation $\alpha / \beta / \gamma / \delta$ (Bitran and Yanasse, 1982)



$\alpha, \beta, \gamma, \delta$ = Z (Zero), C (Constant), NI (Non Increasing), ND (Non Decreasing), G (General)

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Problem modeling
Single-item Lot-Sizing Problems

Capacited single-item lot sizing:

The general case is NP-hard.

Polynomial cases:

Problem	Complexity	References
NI/G/NI/ND	$O(T^4)$ $O(T^2)$	Bitran and Yanasse (1982) Chung and Lin (1988)
NI/G/NI/C	$O(T^3)$	Bitran and Yanasse (1982)
C/Z/C/G	$O(T \log T)$	Bitran and Yanasse (1982)
ND/Z/ND/NI	$O(T)$	Bitran and Yanasse (1982)
G/G/G/C	$\mathbf{O(T^4)}$ $\mathbf{O(T^3)}$	**Florian and Klein (1971)** **Van Hoesel and Wagelmans (1996)**

Problem modeling
Multi-item Lot-Sizing Problem (CLSP)

- **Decision variables:**
  - Y_{it} = 1 if production (setup) of item i in period t, = 0 otherwise
  - X_{it} = Production quantity of item i in period t
  - I_{it} = Inventory of item i at the end of period t
- **Parameters:**
  - d_{it} = Demand for item i in period t
  - v_{it} = Variable production cost per unit of item i in period t
  - s_{it} = Setup cost for item i in period t
  - c_{it} = Inventory holding cost per unit of item i on stock at the end of period t
  - p_{it} = Production time per unit of item i in period t
  - h_t = Available time for production in period t

Problem modeling
Multi-item Lot-Sizing Problem (CLSP)

$$\text{Min} \quad \sum_{i=1}^{N} \sum_{t=1}^{T} \left(s_{it} Y_{it} + v_{it} X_{it} + c_{it} I_{it} \right)$$

subject to :

$$X_{it} + I_{it-1} - I_{it} = d_{it} \qquad \forall i, \forall t$$

$$\sum_{i=1}^{N} p_{it} X_{it} \le h_t \qquad \forall t$$

$$X_{it} \le M_{it} Y_{it} \qquad \forall i, \forall t$$

$$X_{it}, I_{it} \ge 0 \qquad \forall i, \forall t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

CLSP is NP hard in the strong sense (Chen and Thizy, 1990).

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Problem modeling
CLSP with setup times

New parameter: τ_{it} = Fixed setup time of item *i* in period *t*

$$\text{Min} \quad \sum_{i=1}^{N} \sum_{t=1}^{T} \left(s_{it} Y_{it} + v_{it} X_{it} + c_{it} I_{it} \right)$$

$$X_{it} + I_{it-1} - I_{it} = d_{it} \qquad \forall i, \forall t$$

$$\sum_{i=1}^{N} p_{it} X_{it} + \sum_{i=1}^{N} \tau_{it} Y_{it} \le h_t \qquad \forall t$$

$$X_{it} \le M_{it} Y_{it} \qquad \forall i, \forall t$$

$$X_{it}, I_{it} \ge 0 \qquad \forall i, \forall t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

With setup times, checking that a feasible solution exists is already NP-Complete (Trigeiro, Thomas and McClain, 1989).

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Problem modeling
CLSP with backlog

New variable: B_{it} = Backlog of item *i* at the end of period *t*

New parameter: b_{it} = Variable cost per unit of *i* backordered in *t*

$$\text{Min} \quad \sum_{i=1}^{N} \sum_{t=1}^{T} \left(s_{it} Y_{it} + v_{it} X_{it} + c_{it} I_{it} + b_{it} B_{it} \right)$$

$$X_{it} + (I_{it-1} - B_{it-1}) - (I_{it} - B_{it}) = d_{it} \qquad \forall i, \forall t$$

$$\sum_{i=1}^{N} p_{it} X_{it} + \sum_{i=1}^{N} \tau_{it} Y_{it} \le h_t \qquad \forall t$$

$$X_{it} \le M_{it} Y_{it} \qquad \forall i, \forall t$$

$$X_{it}, I_{it}, B_{it} \ge 0 \qquad \forall i, \forall t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

Single-item case solved in $O(T^2)$ with an extension of Wagner-Whitin algorithm (Zangwill, 1969).

Problem modeling
CLSP with lost sales

New variable: L_{it} = Lost sale of item i at the end of period t

New parameter: l_{it} = Variable cost per unit of i lost in t

$$\text{Min} \quad \sum_{i=1}^{N} \sum_{t=1}^{T} \left(s_{it} Y_{it} + v_{it} X_{it} + c_{it} I_{it} + l_{it} L_{it} \right)$$

$$X_{it} + I_{it-1} - (I_{it} - L_{it}) = d_{it} \qquad \forall i, \forall t$$

$$\sum_{i=1}^{N} p_{it} X_{it} + \sum_{i=1}^{N} \tau_{it} Y_{it} \le h_t \qquad \forall t$$

$$X_{it} \le M_{it} Y_{it} \qquad \forall i, \forall t$$

$$X_{it}, I_{it}, L_{it} \ge 0 \qquad \forall i, \forall t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

Single-item case solved in $O(T^2)$ (Aksen, Altinkemer and Chand, 2003).

Problem modeling
CLSP with lost sales

New variable: L_{it} = Lost sale of item i at the end of period t

New parameter: l_{it} = Variable cost per unit of i lost in t

$$\text{Min} \quad \sum_{i=1}^{N} \sum_{t=1}^{T} \left(s_{it} Y_{it} + v_{it} X_{it} + c_{it} I_{it} + l_{it} L_{it} \right)$$

$$X_{it} + I_{it-1} - (I_{it} - L_{it}) = d_{it} \qquad \forall i, \forall t$$

$$\sum_{i=1}^{N} p_{it} X_{it} + \sum_{i=1}^{N} \tau_{it} Y_{it} \le h_t \qquad \forall t$$

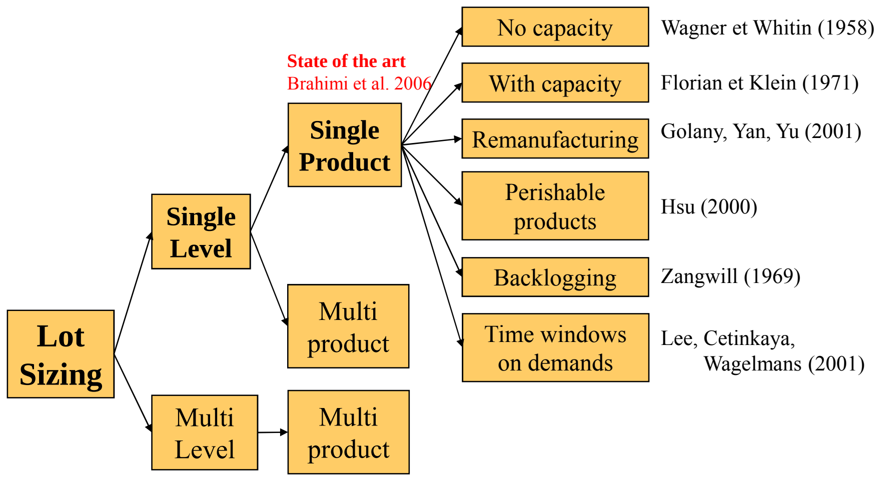
$$X_{it} \le M_{it} Y_{it} \qquad \forall i, \forall t$$

$$X_{it}, I_{it}, L_{it} \ge 0 \qquad \forall i, \forall t$$

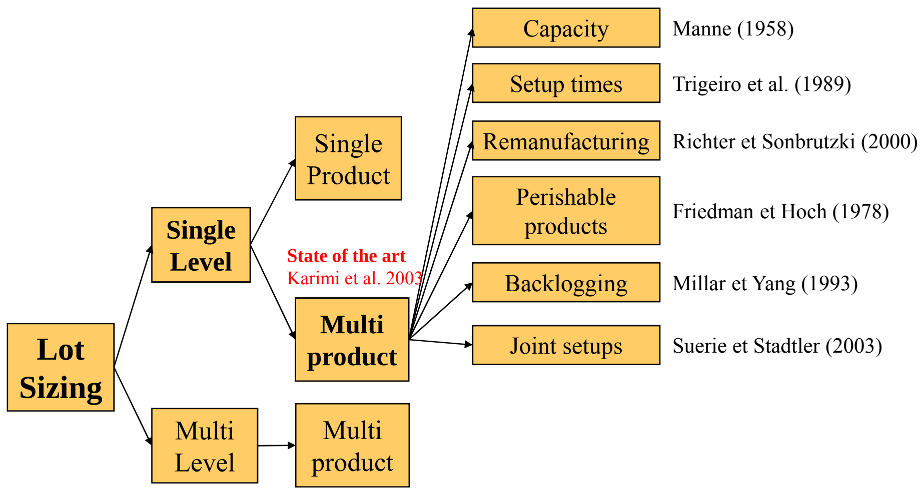
$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

Lagrangian based metaheuristics proposed for the multi-item case (Absi, Detienne and D.-P., 2013).

Classifications



Classifications



Tighter formulations

Multiple formulations were proposed for the single-item and multi-item problems **to improve on the linear relaxation of the aggregate formulation**, and get better solutions with a standard solver.

Disaggregate formulation

Uncapacitated Lot-Sizing Problem (ULSP)

Also called facility location model (Krarup and Bilde, 1977).

- **New variable:** Z_{tk} = quantity produced in period t to cover demand in period k.
- **New parameter:** cd_{tk} = variable production cost plus inventory cost for producing one unit in period t and storing it until period k, i.e.

$$cd_{tk} = v_t + c_t + c_{t+1} + c_{t+2} + \ldots + c_{k-2} + c_{k-1} = v_t + \sum_{u=t}^{k-1} c_u$$

Disaggregate formulation
Uncapacitated Lot-Sizing Problem (ULSP)

$$\text{Min} \quad \sum_{t=1}^{T} s_t Y_t + \sum_{t=1}^{T}\sum_{k=t}^{T} cd_{tk} Z_{tk}$$

$$\sum_{t=1}^{k} Z_{tk} = d_k \qquad \forall k$$

$$Z_{tk} \le d_k Y_t \qquad \forall t, \forall k \ge t$$

$$Z_{tk} \ge 0 \qquad \forall t, \forall k \ge t$$

$$Y_t = \{0,1\} \qquad \forall t$$

Disaggregate formulation
Capacited Lot-Sizing Problem (CLSP)

- **New variable:** Z_{itk} = quantity of item *i* produced in period *t* to cover demand in period *k*.
- **New parameter:** cd_{itk} = variable production cost plus inventory cost for producing one unit of item *i* in period *t* and storing it until period *k*, i.e.

$$cd_{itk} = v_{it} + c_{it} + c_{it+1} + c_{it+2} + \ldots + c_{ik-2} + c_{ik-1} = v_{it} + \sum_{u=t}^{k-1} c_{iu}$$

Disaggregate formulation
Capacited Lot-Sizing Problem (CLSP)

$$\text{Min} \quad \sum_{i=1}^{N}\sum_{t=1}^{T} s_{it}Y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} cd_{itk}Z_{itk}$$

$$\sum_{t=1}^{k} Z_{itk} = d_{ik} \qquad \forall i, \forall k$$

$$\sum_{i=1}^{N}\sum_{k=t}^{T} p_{it}Z_{itk} + \sum_{i=1}^{N} \tau_{it}Y_{it} \le h_t \qquad \forall t$$

$$Z_{itk} \le d_{it}Y_{it} \qquad \forall i, \forall t, \forall k \ge t$$

$$Z_{itk} \ge 0 \qquad \forall i, \forall t, \forall k \ge t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

Shortest Path formulation
Capacited Lot-Sizing Problem (CLSP)

(Eppen and Martin, 1987)

- **New variable:** ZZ_{itk} = fraction of total demand of item *i* for periods *t* through *k* that is produced in period *t* ($\in$ [0,1]).
- **New parameter:** cc_{itk} = variable production cost plus inventory cost for producing in period *t* and storing the demands from periods *t* to *k* of item *i*, i.e.

$$cc_{itk} = \sum_{l=t}^{k} v_{it}d_{il} + c_{it}\sum_{l=t+1}^{k} d_{il} + c_{it+1}\sum_{l=t+2}^{k} d_{il} + \ldots + c_{ik-1}d_{ik}$$

$$cc_{itk} = \sum_{l=t}^{k} v_{it}d_{il} + \sum_{u=t}^{k-1} c_{iu}\sum_{l=u+1}^{k} d_{il}$$

Shortest Path formulation

Capacited Lot-Sizing Problem (CLSP)

$$\text{Min} \quad \sum_{i=1}^{N}\sum_{t=1}^{T} s_{it}Y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} cc_{itk}ZZ_{itk}$$

$$\sum_{k=1}^{T} ZZ_{i1k} = 1 \qquad \forall i$$

$$\sum_{l=1}^{t-1} Z_{ilt-1} = \sum_{k=t}^{T} Z_{itk} \qquad \forall i, \forall t \geq 2$$

$$\sum_{i=1}^{N}\sum_{k=t}^{T} p_{it}\left(\sum_{l=t}^{k} d_{il}\right)ZZ_{itk} + \sum_{i=1}^{N}\tau_{it}Y_{it} \leq h_t \qquad \forall t$$

$$\sum_{k=t}^{T} ZZ_{itk} \leq Y_{it} \qquad \forall i, \forall t$$

$$ZZ_{itk} \geq 0 \qquad \forall i, \forall t, \forall k \geq t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

Solving the single-item case

Wagner-Whitin algorithm

In the uncapacitated case, the **Zero Inventory Order (ZIO) property** is satisfied, i.e. **plans where "$I_{t-1} X_t = 0 \;\; \forall t$" are dominant**

→ Order quantities cover an integer number of periods in dominant plans,

→ ***There is an optimal solution which is a dominant plan,***

→ Wagner Whitin algorithm uses ZIO property,

→ ZIO property often checked when analyzing a new lot-sizing problem.

Solving the single-item case
Wagner-Whitin algorithm

Example.

setup cost	$s_t = 15$	$\forall\ t$
inventory cost	$c_t = 1$	$\forall\ t$
demand	$d_t =$ (4, 8, 6, 7)	

The dominant plans are:

(4, 8, 6, 7)
(4, 8, 13, 0)
(4, 14, 0, 7)
(4, 21, 0, 0)
(12, 0, 6, 7)
(12, 0, 13, 0)
(18, 0, 0, 7)
(25, 0, 0, 0)

Solving the single-item case
Wagner-Whitin algorithm

In general, there will be $2^{(T-1)}$ dominant plans.

By using dynamic programming, the Wagner-Whitin algorithm further reduces the complexity of the search to $T(T+1)/2$, i.e. $O(T^2)$ (Wagner and Whitin, 1958).

Solving the single-item case
Wagner-Whitin algorithm

Example.

setup cost	$s_t = 15$	$\forall\ t$
inventory cost	$c_t = 1$	$\forall\ t$
demand	$d_t = (4, 8, 6, 7)$	

Optimal solution.

$X_1 = 12 \quad X_2 = 0 \quad X_3 = 13 \quad X_4 = 0$

$I_1 = 8 \quad I_2 = 0 \quad I_3 = 7 \quad I_4 = 0$

$Y_1 = 1 \quad Y_2 = 0 \quad Y_3 = 1 \quad Y_4 = 0$

Setup costs:	30
Inventory costs:	15
Total costs:	45

Solving the single-item case
Wagner-Whitin algorithm

In general, there will be $2^{(T-1)}$ dominant plans.

By using dynamic programming, the Wagner-Whitin algorithm further reduces the complexity of the search to $T(T+1)/2$, i.e. $O(T^2)$ (Wagner and Whitin, 1958).

More recently, the complexity has been reduced to $O(T \log(T))$ (Federgruen and Tzur, 1991), (Wagelmans Van Hoesel and Kolen, 1992) , (Aggarwal and Park, 1993).

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Solving the single-item case

Heuristics

Heuristics for single-item problems can be useful to solve more complex problems and to develop heuristics for multi-item problems:

- *Part-Period Balancing*
  Aims at balancing total ordering cost and total holding cost
- *Silver and Meal heuristic*
  Aims at minimizing the cost per period
- *Least Unit Cost*
  Aims at minimizing the total cost per unit of product

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Solving the multi-item case

Lagrangian relaxation

Applying Lagrangian relaxation by relaxing coupling capacity constraints (Trigeiro, Thomas and McClain, 1989).

$$\text{Min} \quad \sum_{i=1}^{N} \sum_{t=1}^{T} \left(s_i Y_{it} + v_i X_{it} + c_i I_{it} \right)$$

subject to :

$$X_{it} + I_{it-1} - I_{it} = d_{it} \qquad \forall i, \forall t$$

$$\sum_{i=1}^{N} \left(p_{it} X_{it} + \tau_{it} Y_{it} \right) \leq h_t \qquad \forall t$$

$$X_{it} \leq M_{it} Y_{it} \qquad \forall i, \forall t$$

$$X_{it}, I_{it} \geq 0 \qquad \forall i, \forall t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

u_t

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Solving the multi-item case

Lagrangian relaxation

Lagrangian relaxation model.

$$\text{Min} \sum_{i=1}^{N} \sum_{t=1}^{T} \left((s_i + u_t \tau_{it}) Y_{it} + (v_i + u_t p_{it}) X_{it} + c_i I_{it} \right) - \sum_{t=1}^{T} u_t h_t$$

subject to :

$$X_{it} + I_{it-1} - I_{it} = d_{it} \quad \forall i, \forall t$$
$$X_{it} \le M_{it} Y_{it} \quad \forall i, \forall t$$
$$X_{it}, I_{it} \ge 0 \quad \forall i, \forall t$$
$$Y_{it} \in \{0,1\} \quad \forall i, \forall t$$

$\rightarrow$ Uncapacitated single-item problems (Wagner Whitin problem) can be solved separately in *O(T logT)*.

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Solving the multi-item case

Lagrangian relaxation

Lagrangian relaxation model.

$$\text{Min} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\boxed{(s_i + u_t \tau_{it})} Y_{it} + \boxed{(v_i + u_t p_{it})} X_{it} + c_i I_{it} \right) - \sum_{t=1}^{T} u_t h_t$$

subject to :

$$X_{it} + I_{it-1} - I_{it} = d_{it} \quad \forall i, \forall t$$
$$X_{it} \le M_{it} Y_{it} \quad \forall i, \forall t$$
$$X_{it}, I_{it} \ge 0 \quad \forall i, \forall t$$
$$Y_{it} \in \{0,1\} \quad \forall i, \forall t$$

Note that Lagrangian production and setup costs are period dependent.

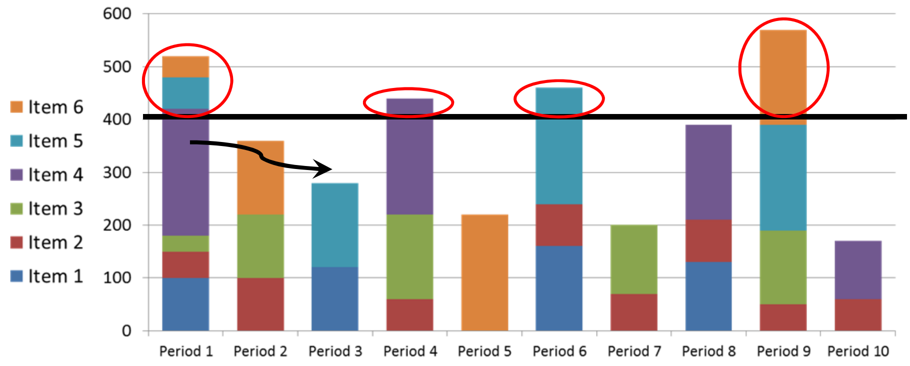
Solving the multi-item case
Lagrangian relaxation heuristic

1. **Solve Lagrangian relaxation model** to determine optimal values of variables X_{it} and Y_{it}.
2. Use solution to **compute Lagrangian lower bound** of optimal solution.
3. Use values of variables obtained in Step 1 to **determine a feasible solution** of original problem (***smoothing heuristic***).
   Update upper bound (best feasible solution).
4. **Update Lagrange multipliers** (subgradient), so that relaxed capacity constraints not satisfied have more chances to be satisfied at next iteration.
   Go to Step 1 if none of the stopping criteria is satisfied (duality gap small enough, step size, number of iterations, …).

Solving the multi-item case
Smoothing (feasibility) heuristic

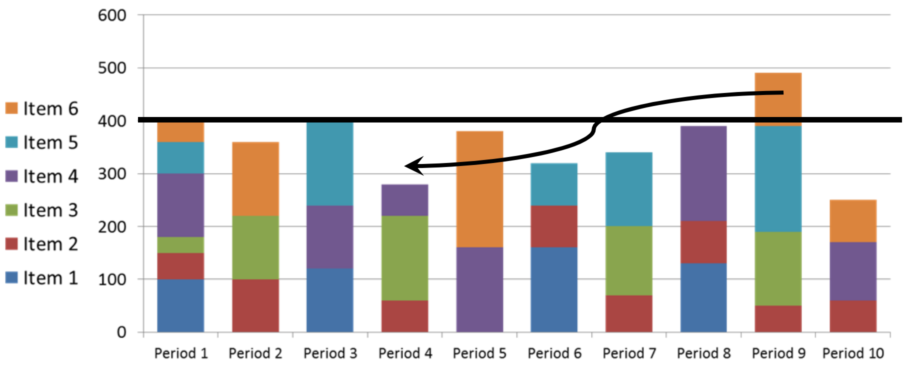
The goal is to ensure feasibility of the plan through forward and backward passes.

Forward pass.



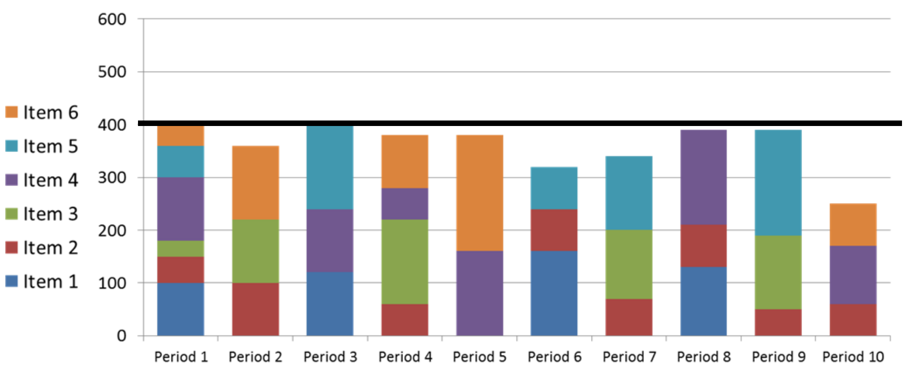
Solving the multi-item case
Smoothing (feasibility) heuristic

Backward pass.



Solving the multi-item case
Smoothing (feasibility) heuristic

Leads to a feasible plan.



Feasibility is not guaranteed (in particular with setup times).

→ Smoothing heuristics can be applied after any heuristic building initial unfeasible solutions.

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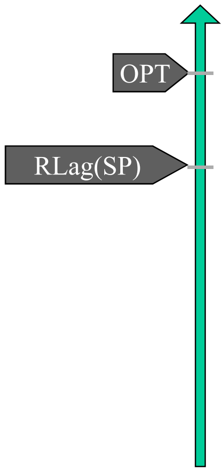
Capacited Lot-Sizing Problem (CLSP)

Comparing lower bounds



$$\text{Min} \quad \sum_{i=1}^{N}\sum_{t=1}^{T} s_{it}Y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} cc_{itk}ZZ_{itk}$$

$$\sum_{k=1}^{T} ZZ_{i1k} = 1 \qquad \forall i \qquad (u_t)$$

$$\sum_{k=t}^{T} Z_{itk} = \sum_{l=1}^{t-1} Z_{ilt-1} \qquad \forall i, \forall t \geq 2 \qquad (v_{it})$$

$$\sum_{i=1}^{N}\sum_{k=t}^{T} p_{it}\left(\sum_{l=t}^{k} d_{il}\right)ZZ_{itk} + \sum_{i=1}^{N} \tau_{it}Y_{it} \leq h_t \qquad \forall t$$

$$\sum_{k=t}^{T} ZZ_{itk} \leq Y_{it} \qquad \forall i, \forall t$$

$$ZZ_{itk} \geq 0 \qquad \forall i, \forall t, \forall k \geq t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

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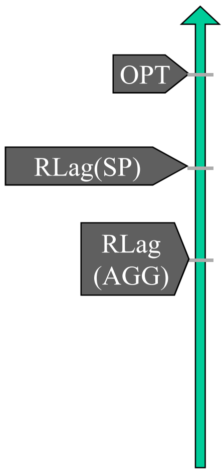
Capacited Lot-Sizing Problem (CLSP)

Comparing lower bounds



$$\text{Min} \quad \sum_{i=1}^{N}\sum_{t=1}^{T}\left(s_{it}Y_{it} + v_{it}X_{it} + c_{it}I_{it}\right)$$

$$X_{it} + I_{it-1} - I_{it} = d_{it} \qquad \forall i, \forall t$$

$$\sum_{i=1}^{N} p_{it}X_{it} \leq c_t \qquad \forall t \qquad (u_t)$$

$$X_{it} \leq M_{it}Y_{it} \qquad \forall i, \forall t$$

$$X_{it}, I_{it} \geq 0 \qquad \forall i, \forall t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

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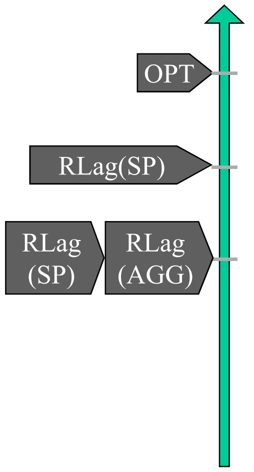
Capacited Lot-Sizing Problem (CLSP)

Comparing lower bounds



$$\text{Min} \quad \sum_{i=1}^{N}\sum_{t=1}^{T} s_{it}Y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} cc_{itk}ZZ_{itk}$$

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$$\sum_{i=1}^{N}\sum_{k=t}^{T} p_{it}\left(\sum_{l=t}^{k} d_{il}\right)ZZ_{itk} + \sum_{i=1}^{N} \tau_{it}Y_{it} \leq h_t \qquad \forall t$$

u_t

$$\sum_{k=t}^{T} ZZ_{itk} \leq Y_{it} \qquad \forall i, \forall t$$

$$ZZ_{itk} \geq 0 \qquad \forall i, \forall t, \forall k \geq t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

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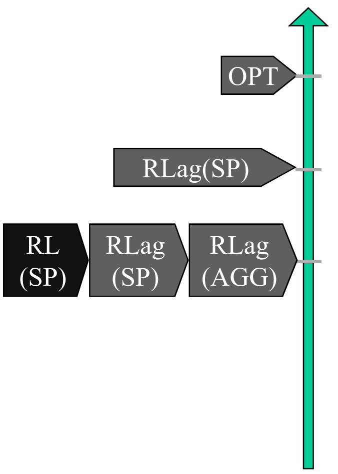
Capacited Lot-Sizing Problem (CLSP)

Comparing lower bounds



$$\text{Min} \quad \sum_{i=1}^{N}\sum_{t=1}^{T} s_{it}Y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} cc_{itk}ZZ_{itk}$$

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$$\sum_{i=1}^{N}\sum_{k=t}^{T} p_{it}\left(\sum_{l=t}^{k} d_{il}\right)ZZ_{itk} + \sum_{i=1}^{N} \tau_{it}Y_{it} \leq h_t \qquad \forall t$$

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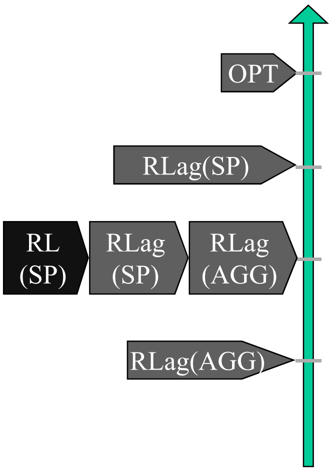
$$ZZ_{itk} \geq 0 \qquad \forall i, \forall t, \forall k \geq t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

Capacited Lot-Sizing Problem (CLSP)

Comparing lower bounds



$$\text{Min} \quad \sum_{i=1}^{N} \sum_{t=1}^{T} \left(s_{it} Y_{it} + v_{it} X_{it} + c_{it} I_{it} \right)$$

$$X_{it} + I_{it-1} - I_{it} = d_{it} \qquad \forall i, \forall t$$

$$\sum_{i=1}^{N} p_{it} X_{it} \leq c_t \qquad \forall t$$

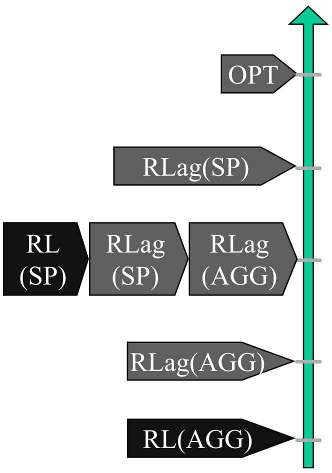
$$X_{it} \leq M_{it} Y_{it} \qquad \forall i, \forall t \qquad \longrightarrow v_{it}$$

$$X_{it}, I_{it} \geq 0 \qquad \forall i, \forall t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

Capacited Lot-Sizing Problem (CLSP)

Comparing lower bounds



$$\text{Min} \quad \sum_{i=1}^{N} \sum_{t=1}^{T} \left(s_{it} Y_{it} + v_{it} X_{it} + c_{it} I_{it} \right)$$

$$X_{it} + I_{it-1} - I_{it} = d_{it} \qquad \forall i, \forall t$$

$$\sum_{i=1}^{N} p_{it} X_{it} \leq c_t \qquad \forall t$$

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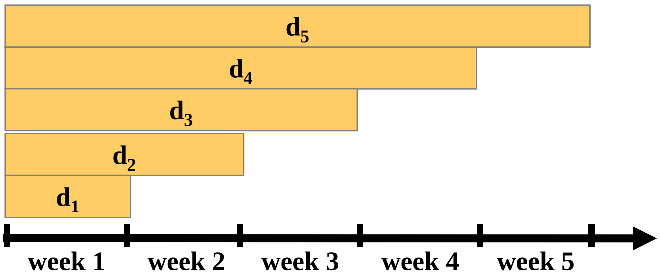
Lot Sizing with Time Windows

In classical lot-sizing problems without time windows, all demands can be processed as early as the first period.



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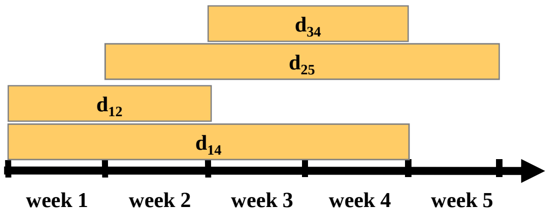
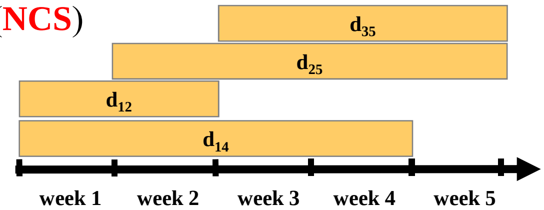
Lot Sizing with Time Windows

In lot-sizing problems with time windows, each demand d_{st} must be processed within the time window $[s,t]$.

Customer Specific (**CS**) (general structure of time windows)



Non Customer Specific (**NCS**) (specific structure of time windows)



Lot Sizing with Time Windows

Complexity for Customer Specific (CS) single-item problem

- Solved using dynamic programming in exponential time in (D.-P., Brahimi, Najid and Nordli, 2002).
- Solved in $O(T^5)$ in (Huang, 2007)

Complexity for Non Customer Specific (NCS) single-item problem

- Exponential time algorithm for CS problem runs in $O(T^4)$ for NCS problem (D.-P., Brahimi, Najid and Nordli, 2002).
- Improved to $O(T^2)$ in (Wolsey, 2006).
- Generalization to early productions, backlogs and lost sales solved in $O(T^2)$ in (Absi, Kedad-Sidhoum and D.-P., 2011).

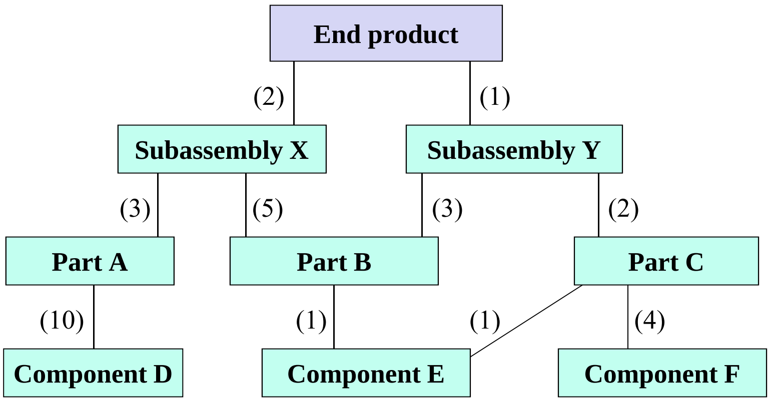
Lagrangian relaxation heuristics proposed in (Brahimi, D.-P. and Najid, 2006) **for CS and NCS multi-item problems.**

Green Lot Sizing

Lot Sizing with carbon emissions

- Recent research on lot sizing is concerned with considering new environmental constraints.
- A **global carbon emission constraint** is considered in (Benjaafar, Li and Daskin, 2010).
  → Acts as a capacity constraint.
- Four types of carbon emission constraints are proposed and analyzed in the single-item case in (Absi, D.-P., Kedad-Sidhoum, Penz and Rapine, 2013).
  → Theses constraints do not limit lot sizes, but **limit the carbon emission per unit of product**.
  → *See presentation of S. Kedad-Sidhoum.*

Multi-Level (Multi-Stage) Lot Sizing



Independent demands (for end products) and **dependent demands**

Multi-Level Lot Sizing

New parameter g_{ij}: Number of items i necessary to produce one unit of item j (*gozinto factor*).

$$\text{Min} \quad \sum_{i=1}^{N} \sum_{t=1}^{T} \left(s_{it} Y_{it} + v_{it} X_{it} + c_{it} I_{it} \right)$$

$$X_{it} + I_{it-1} - I_{it} = d_{it} + \sum_{j=1}^{N} g_{ij} X_{jt} \qquad \forall i, \forall t$$

$$\sum_{i=1}^{N} p_{it} X_{it} \qquad \forall t$$

Independent demand emand

$$X_{it} \le M_{it} Y_{it} \qquad \forall i, \forall t$$

$$X_{it}, I_{it} \ge 0 \qquad \forall i, \forall t$$

$$Y_{it} \in \{0,1\} \qquad \forall i, \forall t$$

New inventory balance equation *becomes a coupling constraint.*

Multi-Level Lot Sizing

See presentation of J.-P. Casal (FuturMaster).

Various approaches have been proposed:

- Lagrangian relaxation (Tempelmeier and Derstroff, 1996),
- MIP-based heuristics (used in general for complex lot-sizing problems).

MIP-based heuristics for Lot Sizing

- **Iterative approaches.**
- Solve at each stage a **reduced mixed integer problem**.
- By reducing the number of binary variables and the number of constraints.
- **Various decompositions** can be used::
  - An horizon-oriented decomposition,
  - A product-oriented decomposition,
  - A resource-oriented decomposition,
  - A process-oriented decomposition, etc.

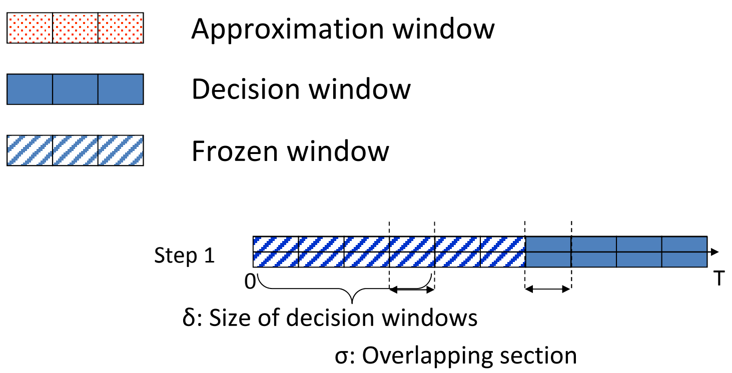
MIP-based heuristics for Lot Sizing

Several variants:

- **Relax-and-Fix, Fix-and-Relax** (Kelly 2002, Clark 2003, Mercé and Fontan 2003, , Stadtler 2003, Pochet and Van Vyve 2004, Absi and kedad-Sidhoum 2007, Federgruen et al. 2007, Seeanner et al. 2013),
- **Fix-and-Optimize** (Sahling et al. 2009, S Helber, F Sahling 2010, Lang and Shen 2011, James and Almada-Lobo 2011).

MIP-based heuristics for Lot Sizing

Example of an horizon-oriented decomposition (Absi and Kedad-Sidhoum 2007) – Relax-and-Fix



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Integrating Lot-Sizing decisions and other types of decisions

More and more researchers are studying the integration of lot-sizing decisions with decisions taken at other levels or other stages in the supply chain.

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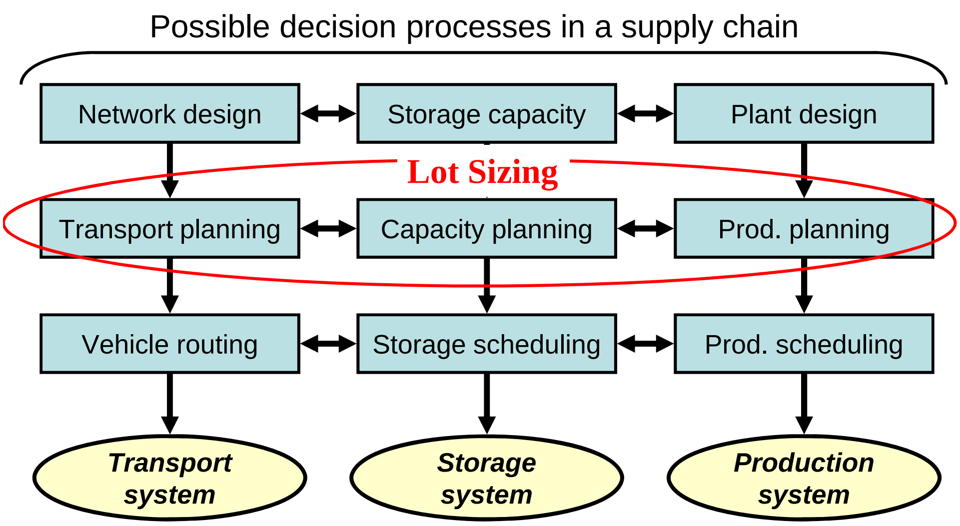
Integrating Lot-Sizing decisions and other types of decisions



Integrating Lot-Sizing decisions and other types of decisions

Variables at different levels/stages are often of different nature, i.e. no longer pure continuous or integer optimization problems

→ Makes the integration particularly complex and changes the nature of the problems.

Objectives at different levels/stages may be of different nature, e.g. cost minimization vs. time criteria.

Integrating Lot-Sizing and Cutting-Stock decisions

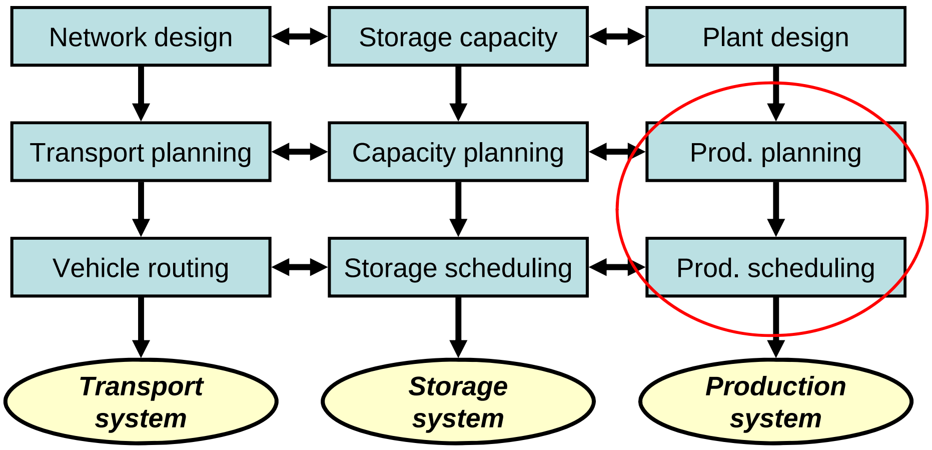
Column Generation approaches are used in (Nonås and Thorstenson, 2000) and (Nonås and Thorstenson, 2008) to solve a combined lot-sizing and cutting-stock problem.

(Gramani and França, 2006) analyzes the trade-off in industrial problems, where trim loss, storage and setup costs are minimized. The problem is solved using a network shortest path formulation.

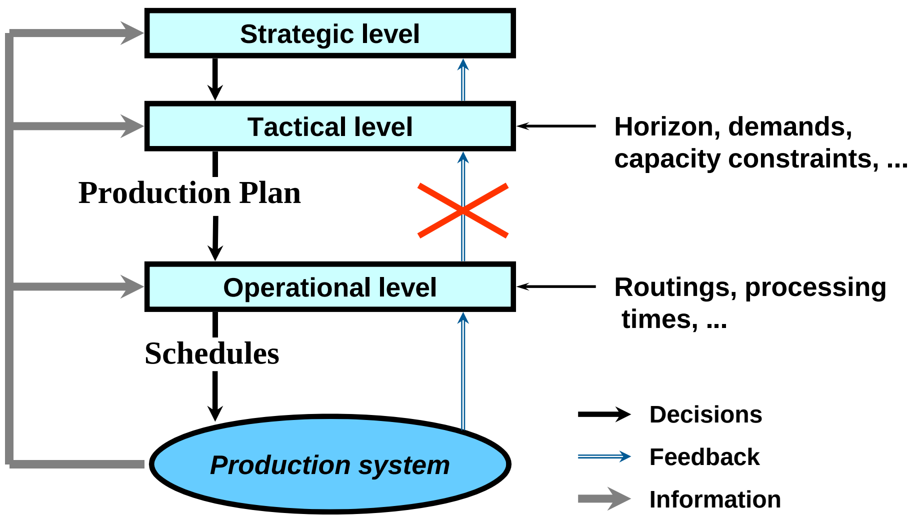
(Gramani, França and Arenales, 2009) proposes a Lagragian relaxation heuristic.

Heuristics are also proposed in (Poltroniere, Poldi, Toledo and Arenales, 2008)

Integrating Lot-Sizing and Scheduling decisions



Integrating Lot-Sizing and Scheduling decisions



Integrating Lot-Sizing and Scheduling decisions

Planning and scheduling can hardly be treated *simultaneously*

Production planning (Lot Sizing)	**How much** (continuous variables)
↓	
Scheduling	**When and How** (discrete variables)

Aggregate (necessary) capacity constraints are used

→ No ***actual*** schedule to satisfy the plan

→ Delays, work-in-process inventories

Integrating Lot-Sizing and Scheduling decisions

- Integration of production planning and detailed scheduling (Lasserre 1989, D-P. and Lasserre 1994 and 2002)
  - Multi-item lot-sizing problem,
  - Combined with job-shop scheduling problem.
- Scientific challenges:
  - Multi-item lot-sizing problem with complex capacity constraints,
  - Or job-shop scheduling problem where processing times are variables.
- Practical challenges (e.g. Renault and “25% rule” in 1996).

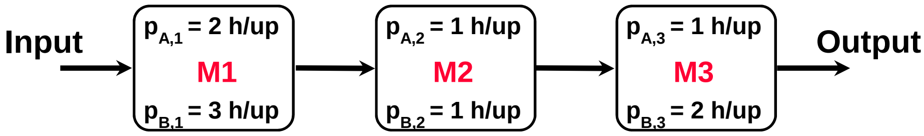
Integrating Lot-Sizing and Scheduling decisions – *A simple example*

Two products A and B

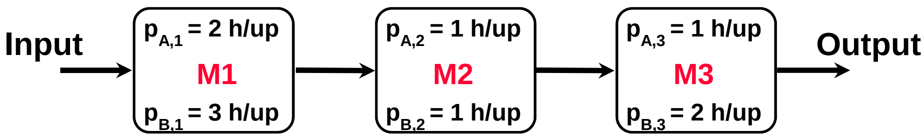
Three machines M1, M2 and M3

Length of a period: 60 h

Quantities to be produced: XA and XB



Integrating Lot-Sizing and Scheduling decisions – *A simple example*



Machine constraints:

M1: $2\ XA + 3\ XB \leq 60$

M2: $XA + XB \leq 60$

M3: $XA + 2\ XB \leq 60$

→ XA = 10 and XB = 10 are feasible

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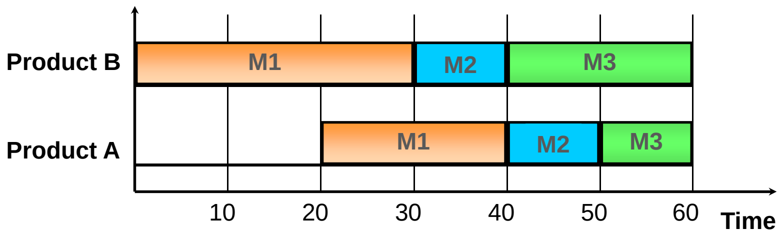
Integrating Lot-Sizing and Scheduling decisions – *A simple example*



Schedule of the operations in classical planning (M.R.P.)

→ Operations are not **sequenced**

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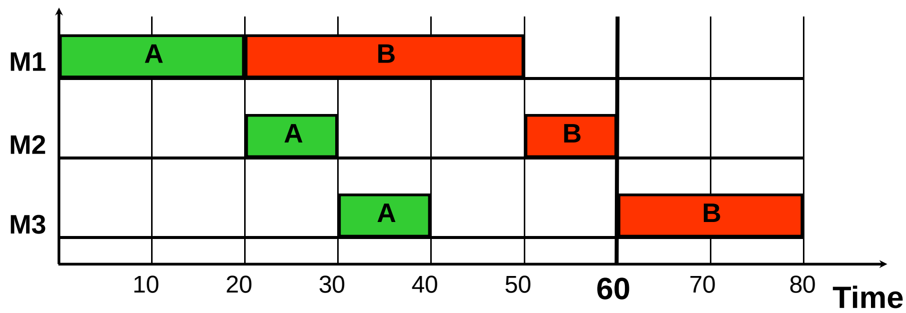
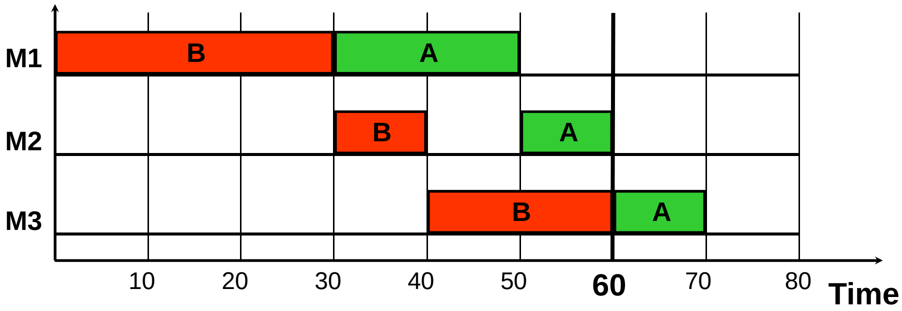
Integrating Lot-Sizing and Scheduling decisions – *A simple example*



Product A sequenced before product B

Integrating Lot-Sizing and Scheduling decisions – *A simple example*



Product B sequenced before product A

Integrating Lot-Sizing and Scheduling decisions

$$\min \sum_{i,l} c_i^p X_{il} + \sum_{i,l} (c_i^{inv} I_{il}^+ + c_i^{back} I_{il}^-) \quad (7)$$

$$I_{il}^+ - I_{il-1}^- = I_{il-1}^+ - I_{il-1}^- + X_{il} - \sum_{j \in \mathscr{DS}(i)} g_{ij} X_{jl+L_j} - D_{il} \quad \forall i, \forall l \quad (8)$$

Planning (Lot-Sizing) problem

$$t_{o'} \geqslant t_o \quad (9)$$

$$t_{o'} \geqslant t_o + p_o^u X_{i(o)l(o)}$$

or

$$t_o \geqslant t_{o'} + p_{o'}^u X_{i(o')l(o')} \quad \forall (o, o') \in \mathscr{S}(y) \quad (10)$$

$$t_o + p_o^u X_{i(o)l(o)} \leqslant \sum_{l=1}^{l(o)} c_l \quad \forall o \in \mathscr{L} \quad (11)$$

$$t_o + p_o^u X_{i(o)l(o)} \geqslant \sum_{l=1}^{l(o)-1} c_l \quad \forall o \in \mathscr{L} \quad (12)$$

$$t_o \geqslant \sum_{l=1}^{l(o)-L_{i(o)}} c_l \quad \forall o \in \mathscr{F} \text{ such that } L_{i(o)} > 0 \quad (13)$$

$$X_{il}, X_{il}^-, \quad (14)$$

Scheduling problem

$$t_o \geqslant 0 \quad (15)$$

Integrating Lot-Sizing and Scheduling decisions

- A **two-level iterative procedure** has been used to solve the problem (Lasserre 1989, D.-P. and Lasserre 1994, 2002, Roux, D.-P. and Lasserre 1999)

Comparison between **feasible** production plans obtained with aggregate model and integrated model with one-pass and iterative procedures

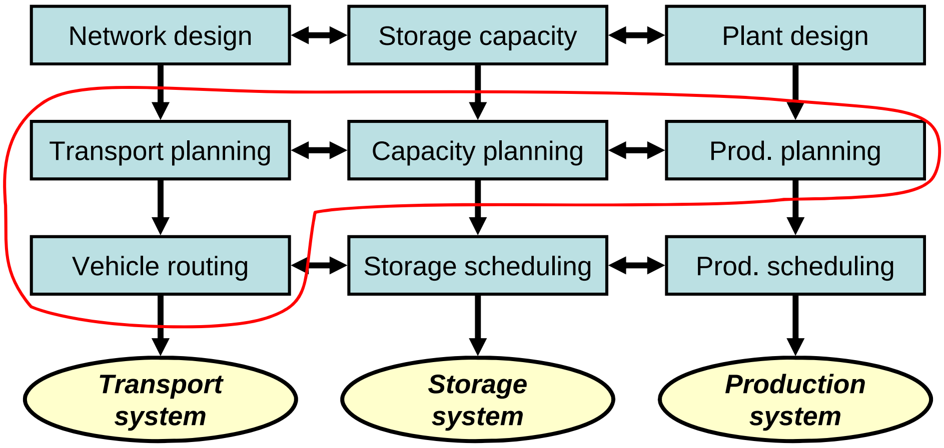
Problem	Total inventory and backlog costs		
	Aggregate	*One-pass*	*Iterative*
1	7538	5248	**4347**
2	17036	12638	**12246**
3	6318	3913	**2651**
4	2457	883	**133**
5	3318	482	**220**
6	698	0	**0**

Integrating Lot-Sizing and Scheduling decisions

- A **two-level iterative procedure** has been used to solve the problem (Lasserre 1989, D.-P. and Lasserre 1994, 2002, Roux, D.-P. and Lasserre 1999)
- More recently an **integrated approach** has been proposed in (Wolosewicz, D.-P. and Aggoune, 2008)
  → Pursued in PhD thesis of Edwin Gomez for multi-level lot-sizing problems in a supply chain
- Novel formulation based on **graph representation** of scheduling problem, where *each path corresponds to a capacity constraint*.
  → Exponential number of capacity constraints
  → **Lagrangian relaxation approach** where violated paths are inserted one by one with positive Lagrangian multiplier

Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

- Integrated optimization of **production**, **distribution** and **inventory** decisions.

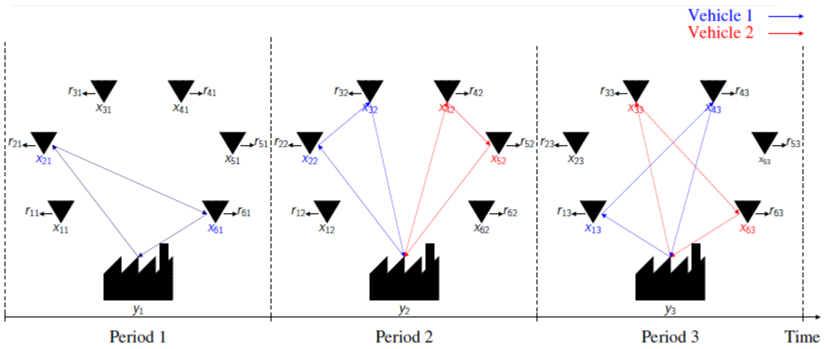


Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

- Integrated optimization of **production**, **distribution** and **inventory** decisions.
- The PRP simultaneously optimizes production, inventory and routing so that final demands of customers and inventory limits in production facility and retailers are satisfied, while minimizing all types of costs.

Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

- Decide **when** and **how much** to produce, **when**, **how much** and **how** to transport in order to satisfy customer **demands** over a discrete time horizon



Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

Literature review

- Few papers address the PRP.
- The problem of integrating production and routing decisions was introduced by (Chandra, 1993).
- Most authors used heuristic methods to solve the problem of integrating production planning and vehicle routing (Boudia et al., 2007), (Bard et al., 2009), (Adulyasak et al. 2012), (Absi et al., 2013).
- Very few authors used exact methods (Archetti et al., 2011).
- More authors addressed the Inventory Routing Problem (IRP), which does not consider production.

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Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

Mathematical model

- Minimize production, inventory and routing costs subject to:
  - Inventory balance constraints for retailers and production facility,
  - Inventory capacity constraints for retailers and production facility,
  - Production constraints,
  - Vehicle capacity constraints,
  - Routing constraints.

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Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

Solution approach (Absi, Archetti, Feillet and D.-P. 2012)

- ***Iterative two-phase approach.***
- **Routing costs** incurred when visiting a customer at a given period **are approximated** and denoted SC_{it}.
  → Initial model can then be transformed into a lot-sizing model that optimizes production and inventory levels.
- Distribution costs only interfere with the lot-sizing model through the setup costs SC_{it}.
- First phase is called **Lot-sizing Problem** (SC_{it}).
- Using the solution obtained in first phase, routing decisions are taken in second phase, called **Routing Problem** (γ_{vit}).
  → Corresponds to solving the vehicle routing problem once the γ_{vit} variables are fixed.

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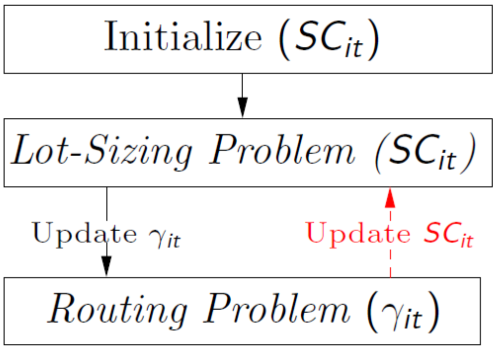
Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

Solution approach (general scheme)



Diversification mechanism: SC_{it} multiplied by coefficient (route length) when customer i belongs to existing route at period t.

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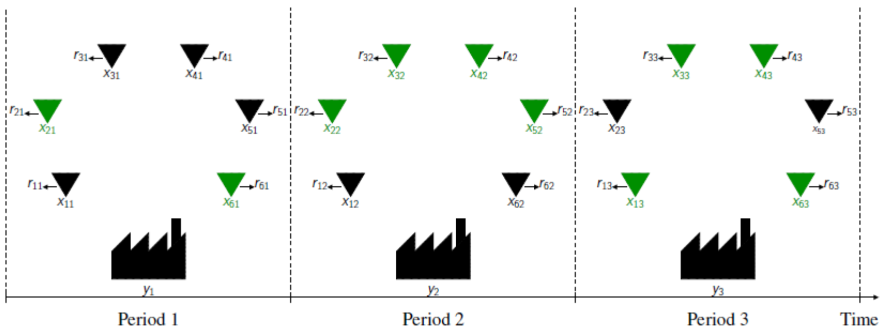
Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

Lot-sizing phase

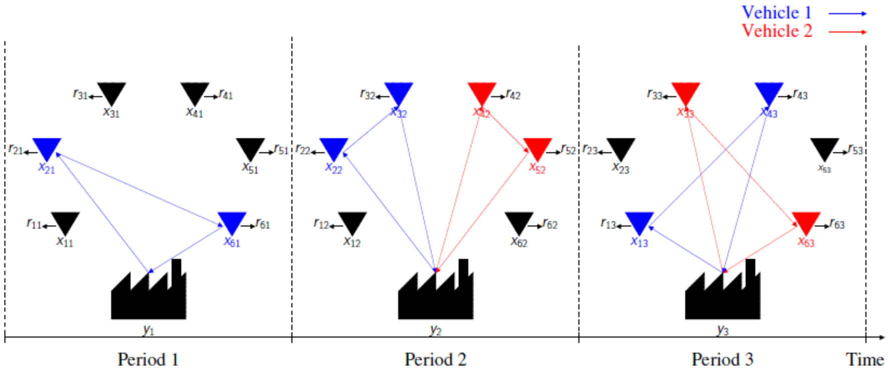
- Optimize production plan with approximated routing costs SC_{it} (setup costs),
- Decide **when** (γ_{vit}) and **how much** to produce, **when** and **how much** to transport in order to satisfy customer demands.



Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

Routing phase

- Decide **how** to transport goods in order to satisfy customers and vehicles capacities (**Vehicle Routing Problem**).



Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

Computational experiments

- Stops after 20 iterations.
- When solution not improved for 5 iterations, diversification mechanism is used.
- Comparison with heuristics of (Archetti et al., 2011) (H) and (Adulyasak et al., 2012) (Op-ALNS).

Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

Computational experiments

Instance set	A1	A2	A3
No. of instances	480	480	480
No. of periods	6	6	6
No. of retailers	14	50	100
No. of trucks	1	∞	∞
Demand	C	C	C
Production capacity	∞	∞	∞
Plant inventory capacity	∞	∞	∞
Retailer inventory capacity	C	C	C
Initial inventory at plant	0	0	0
Initial inventory at retailers	V	V	V
Vehicle capacity	C	C	C

V - Varying, C - Constant, ∞ - Unlimited

Table: Overview of the benchmark instances (Archetti et al. 2011)

Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

Computational experiments

Class Type	Descriptions
Class I 1-24	Standard instances
Class II 25-48	High production unit cost, $\times 10$
Class III 49-72	Large transportation costs, $\times 5$
Class IV 73-96	No retailer inventory costs

Table: Descriptions of instances classes (Archetti et al. 2011)

Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

Computational results

Average gaps to best solutions for 480 instances of set A1.

Classes	IM	H	ALNS
1	0,13%	2,13%	1,65%
2	0,02%	0,30%	0,36%
3	0,71%	3,43%	7,60%
4	0,07%	0,88%	0,93%
All	0,23%	1,68%	2,64%

Integrating Lot Sizing and Routing
Production Routing Problem (PRP)

Computational results

Average gaps to best solutions for 480 instances of set A2.

Classes	IM	H	ALNS
1	0,04%	1,89%	0,98%
2	0,02%	0,35%	0,14%
3	0,26%	2,66%	2,66%
4	0,04%	1,17%	0,13%
All	0,09%	1,52%	0,98%

Average gaps to best solutions for 480 instances of set A3.

Classes	IM	H	ALNS
1	0,06%	2,06%	0,82%
2	0,19%	0,32%	0,29%
3	0,23%	2,55%	2,53%
4	0,18%	1,19%	0,26%
All	0,16%	1,53%	0,98%

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Conclusions

- Lot sizing is (again…) an active field of research.
- Numerous topics were not discussed in this presentation such as: sequence-dependent setup times, joint setups, inventory bounds, stochastic lot sizing, various solution approaches (Column Generation, metaheuristics, …), …
- A lot of research remains to be done on the interface between lot sizing and other problems to define, in particular with industry, relevant combined problems.