Non-Robust Strong Knapsack Cuts for Capacitated Location-Routing and Related Problems

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Capacitated Location-Routing Problem (CLRP)

Data

- I — set of potential depots with opening costs f_i and capacities W_i, $i \in I$
- J — set of customers with demands d_j, $j \in J$
- Set of edges $E = E_J \cup E_{IJ}$: $E_J = J \times J$, $E_{IJ} = I \times J$
- c_e — transportation cost of edge $e \in E \cup F$
- An unlimited set of vehicles with capacity Q.

The problem

- Decide which depots to open
- Assign every client to an open depot subject to depot capacity
- For every depot, divide assigned clients into routes subject to vehicle capacity
- Minimize the total depot opening and transportation cost

CLRP: an illustration

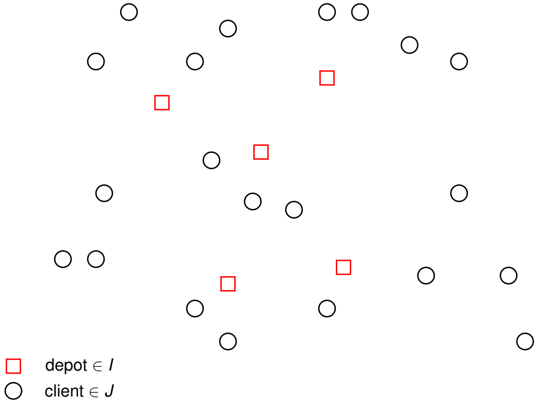


Figure: LRP instance: $G = (I \cup J, E_J \cup E_{IJ})$

CLRP: a solution

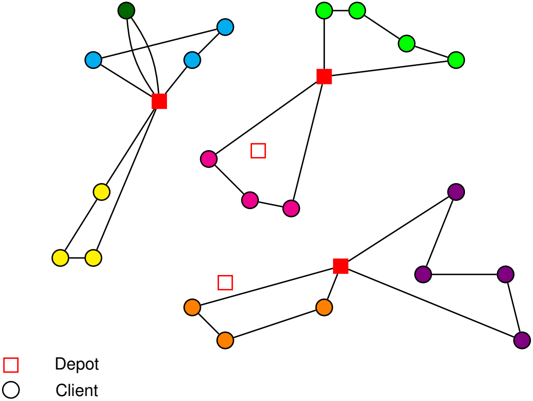


Figure: Location of depots must be jointly decided with vehicle routing.

Literature on the CLRP

- A combination of two central OR problems
- ≈3700 papers in Google Scholar with both "location" and "routing" in the title

Some recent works on the standard CLRP

- [Belenguer et al. 2011] — important valid inequalities & Branch-and-Cut
- [Baldacci et al. 2011] — exact enumeration & column generation approach
- [Contardo et al. 2014][1] — state-of-the-art exact algorithm
- [Arnold and Sörensen 2021; Schneider and Löffler 2019] — state-of-the-art heuristics
- [Schneider and Drexl 2017] — the latest survey

[1] Claudio Contardo, Jean-François Cordeau, and Bernard Gendron (2014). "An Exact Algorithm Based on Cut-and-Column Generation for the Capacitated Location-Routing Problem". In: *INFORMS Journal on Computing* 26.1, pp. 88–102.

Our study

- Recently, large improvement in exact solution of classic VRP variants [Pecin et al. 2017] [Pecin et al. 2017] [Pessoa et al. 2018] [Sadykov et al. 2021]
- A generic Branch-Cut-and-Price VRP solver [Pessoa et al. 2020][2] incorporates all recent advances

  vrpsolver.math.u-bordeaux.fr

- This solver can be applied to the LRP, but problem-specific cuts are necessary for obtaining the state-of-the-art performance
- We propose a new family of non-robust cuts, which is shown to be useful for the LRP and some related problems

[2] Artur Pessoa, Ruslan Sadykov, Eduardo Uchoa, and François Vanderbeck (2020). “A Generic Exact Solver for Vehicle Routing and Related Problems”. In: *Mathematical Programming* 183, pp. 483–523.

Formulation

- λ_r^i, $i \in I$, $r \in R_i$, equals 1 iff route r is used for depot i
- a_e^r, $e \in E_J \cup E_{IJ}$, $r \in \cup_{i \in I} R_i$, equals 1 iff edge e is used by r
- b_j^r, $j \in J$, $r \in \cup_{i \in I} R_i$, equals 1 iff client i is served by route r
- y_i, $i \in I$, equals 1 iff route depot i is open
- z_{ij}, $i \in I$, $j \in J$, equals 1 iff client j is assigned to depot i

$$\min \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{r \in R_i} \sum_{e \in E \cup F} c_e a_e^r \lambda_r^i$$

$$\sum_{i \in I} z_{ij} = 1, \qquad \forall\, j \in J,$$

$$\sum_{r \in R_i} b_j^r \lambda_r^i = z_{ij}, \qquad \forall\, i \in I,\ j \in J$$

$$\sum_{j \in J} d_j z_{ij} \leq W_i y_i, \qquad \forall\, i \in I,$$

$$z_{ij} \leq y_i, \qquad \forall\, i \in I,\ j \in J,$$

$$(z, y, \lambda) \in \{0, 1\}^K$$

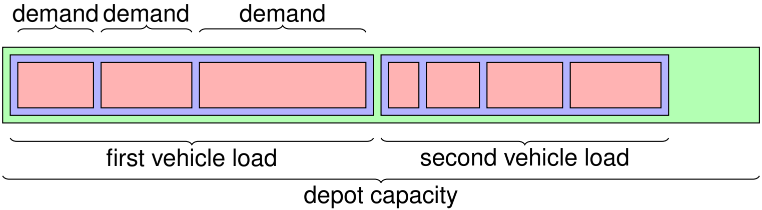
Nested knapsack structure

- λ_r^i, $i \in I$, $r \in R_i$, equals 1 iff route r is used for depot i
- b_j^r, $j \in J$, $r \in \cup_{i \in I} R_i$, equals 1 iff client i is served by route r
- z_{ij}, $i \in I$, $j \in J$, equals 1 iff client j is assigned to depot i

$$\sum_{j \in J} d_j z_{ij} \leq W_i, \quad \forall\, i \in I,$$

$$\Downarrow$$

$$\sum_{r \in R_i} \left(\sum_{j \in J} \sum_{e \in \delta(j)} b_j^r d_j \right) \lambda_r^i \leq W_i, \quad \forall\, i \in I$$



Master knapsack polytope

$$\mathcal{P}_{\mathrm{MKP}}(W) = \mathrm{conv}\left\{ t \in \mathbb{Z}_+^W : \sum_{q=1}^{W} q\, t_q \le W \right\}.$$

Theorem ([Aráoz 1974][3])

Each non-trivial facet of $\mathcal{P}_{MKP}(W)$ can be described by an inequality of format $\xi t \le 1$ such that $\xi \in \mathbb{R}_+^W$ is an extreme point of the following system of linear constraints:

$$\begin{aligned} \xi_1 = 0, \quad \xi_W = 1, & \\ \xi_q + \xi_{W-q} = 1, & \qquad \forall 1 \le q \le W/2, \\ \xi_q + \xi_{q'} \le \xi_{q+q'}, & \ \forall q + q' \le W. \end{aligned}$$

[3] J. Aráoz (1974). "Polyhedral neopolarities". PhD thesis. University of Waterloo, Department of Computer Science.

Route Load Knapsack Cuts (RLKC)

$\theta_q^i \in \mathbb{Z}_+$ — # of routes in R_i with a total load of *exactly* q units.

Theorem

$\xi t \leq 1$ defines a non-trivial facet of $\mathcal{P}_{MKP}(W_i)$ if and only if $\xi\theta \leq y_i$ defines a non-trivial facet of $\text{conv}\left\{(\theta^i, y_i) \in \mathbb{Z}_+^{W_i} \times \{0,1\} : \sum_{q=1}^{W_i} q\,\theta_q \leq W_i y_i\right\}$.

Definition

Given a depot $i \in I$ and a vector $\xi \in \mathbb{R}_+^{W_i}$ satisfying $\xi_1 = 0$, $\xi_{W_i} = 1$, and $\xi_q + \xi_{q'} \leq \xi_{q+q'}$, $\forall q + q' \leq W_i$, the inequality

$$\sum_{q=1}^{W_i} \xi_q\,\theta_q^i \leq y_i$$

is known as a *Route Load Knapsack Cut* (RLKC).

Theorem

A Route Load Knapsack Cut is valid for the location-routing formulation.

Separation of RLKCs by Chvàtal-Gomory rounding

For any depot $i \in I$ and any multiplier $\beta \in \mathbb{R}$ such that $\beta \geq 1/W_i$, the constraint

$$\sum_{q=1}^{W_i} \frac{\lfloor \beta\, q \rfloor}{\lfloor \beta\, W_i \rfloor} \theta_q^i \leq y_i$$

is a Route Load Knapsack Cut: superadditivity follows from $\lfloor r \rfloor + \lfloor r' \rfloor \leq \lfloor r + r' \rfloor$ for all $r, r' \in \mathbb{R}_+$.

Separation by enumeration

We consider all multipliers $\beta = p/q$, such that $\bar{\theta}_q > 0$ and $p = 1, \ldots, q - 1$.

Separation of $1/k-$facets

Definition

A master knapsack facet $\xi x \leq 1$ is called a $1/k$-facet if k is the smallest possible integer such that

$$\xi_q \in \{0/k, 1/k, 2/k, \ldots, k/k\} \cup \{1/2\}.$$

Separation by enumeration

We enumerate all possible $1/6-$, $1/8-$, and $1/10-$inequalities using the algorithm from [Chopra et al. 2015][4].

Remark

An $1/k'$-inequality is also an $1/k$-inequality if k' is a divisor of k. So, $1/2$-, $1/3$-, $1/4$-, and $1/5$-inequalities are also separated.

[4] Sunil Chopra, Sangho Shim, and Daniel E. Steffy (2015). "A few strong knapsack facets". In: *Modeling and Optimization: Theory and Applications*. Ed. by Boris Defourny and Tamás Terlaky. Cham: Springer International Publishing, pp. 77–94.

Route Load Knapsack Cuts: an example

Data (depot index i is omitted)

$Q = 70$, $W = 140$, $\bar{\theta}_{38} = 1/14$, $\bar{\theta}_{53} = 1/2$, $\bar{\theta}_{65} = 16/14$, $\bar{\theta}_{70} = 1/2$, and $\bar{y} = 277/280$.

Best RLKC obtained by rounding

Multiplier $\beta = 1/53$ gives a RLKC with violation of ≈ 0.082:

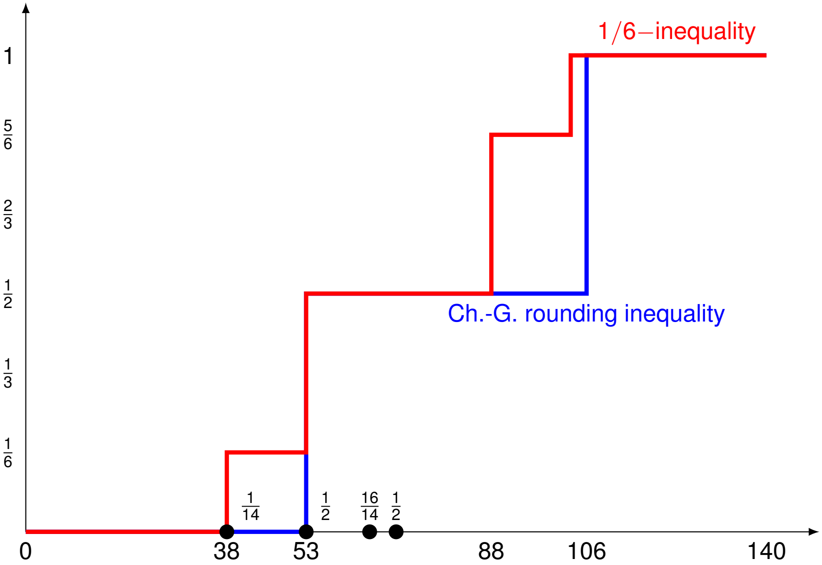
$$\sum_{q=53}^{105} \frac{1}{2}\theta_q + \sum_{q=106}^{140} \theta_q \leq y$$

A better facet-defining $1/6-$inequality

$$\sum_{q=38}^{52} \frac{1}{6}\theta_q + \sum_{q=53}^{87} \frac{3}{6}\theta_q + \sum_{q=88}^{102} \frac{5}{6}\theta_q + \sum_{q=103}^{140} \frac{6}{6}\theta_q \leq y$$

with violation ≈ 0.094.

Route Load Knapsack Cuts: an example



RLKCs and the pricing: label domination

- Pricing problem: Resource Constrained Shortest Path
- It is solved by a labelling algorithm, each label L is $(j^L, \bar{c}^L, q^L)$
- Dominance relation

$$L \succ L' \quad \text{if } j^L = j^{L'},\ \bar{c}^L \leq \bar{c}^{L'},\ q^L \leq q^{L'}$$

- Let $\bar{\mu}(q)$ be the contribution of RLKCs to the reduced cost of a route variable with load q
- The same dominance relation is still valid, as $\bar{\mu}(q)$ is non-decreasing:

$$\bar{c}^L \leq \bar{c}^{L'}, q^L \leq q^{L'} \quad \Rightarrow \quad \bar{c}^L + \bar{\mu}(q^L) \leq \bar{c}^{L'} + \bar{\mu}(q^{L'}).$$

RLKCs and the pricing: completion bounds

- A completion bound $B(j, \overleftarrow{\mathcal{L}})$ is valid if $\bar{c}^{\vec{L}} + \bar{\mu}(q^{\vec{L}}) + B(j, \overleftarrow{\mathcal{L}})$ gives a lower bound on the total reduced cost of paths obtained by concatenation of $\vec{L}$ and any $\overleftarrow{L} \in \overleftarrow{\mathcal{L}}$ at node j.

- A weak completion bound

$$B_1(j, \overleftarrow{\mathcal{L}}) = \min_{\overleftarrow{L} \in \overleftarrow{\mathcal{L}}} \left\{ \bar{c}^{\overleftarrow{L}} \right\}$$

- A tighter completion bound

$$B_2(j, \overleftarrow{\mathcal{L}}) = \min_{\overleftarrow{L} \in \overleftarrow{\mathcal{L}}} \left\{ \bar{c}^{\overleftarrow{L}} + \bar{\mu}(q^{\overleftarrow{L}}) \right\}$$

  is valid due to super-additivity of $\bar{\mu}(q)$.

- Completion bounds are used to speed-up labels concatenation and prune labels by bound

Rounded Capacity Cuts (RCC)[5]

Given a subset of clients $C \subset J$,

$$\sum_{i \in I} \sum_{r \in R_i} \sum_{e \in \delta(C)} a_e^r \lambda_r^i \geq 2 \cdot \left\lceil \frac{\sum_{i \in C} d_i}{Q} \right\rceil .$$

Separation

Heuristic algorithms by [Lysgaard et al. 2004]: connected components, max-flow based, greedy construction, local search on previously separated cuts (CVRPSEP reimplemented by us)

[5] G. Laporte and Y. Nobert (1983). "A branch and bound algorithm for the capacitated vehicle routing problem". In: *Operations-Research-Spektrum* 5.2, pp. 77–85.

Depot Capacity Cuts (DCC)[6]

If a subset of clients $C \subset J$ cannot be served by a subset of depots $S \subset I$,

$$\sum_{j \in C} d_j > \sum_{i \in S} W_i,$$

then at least one vehicle from a depot $i \in I \setminus S$ should visit C:

$$\sum_{i \in I \setminus S} \sum_{r \in R_i} \sum_{e \in \delta(C)} a_e^r \lambda_r^i \geq 2.$$

Separation

A greedy construction heuristic starting from different seed vertices

[6] José-Manuel Belenguer, Enrique Benavent, Christian Prins, Caroline Prodhon, and Roberto Wolfler Calvo (2011). "A Branch-and-Cut method for the Capacitated Location-Routing Problem". In: *Computers & Operations Research* 38.6, pp. 931 –941.

COVer inequalities for depots (COV)

Given a subset $J' \subset J$ of customers, such that $\sum_{j \in J'} d_j > W_i$, the following inequality is valid

$$\sum_{j \in J'} z_{ij} \leq (|J'| - 1) y_i.$$

Separation

We solve the MIP for each $i \in I$ such that $\bar{y}_i > 0$

$$\min \sum_{j \in J} (\bar{y}_i - \bar{z}_{ij}) \, w_j$$

$$\sum_{j \in J} d_j w_j \geq W_i + 1,$$

$$w_j \in \{0, 1\}, \qquad \forall \, j \in J,$$

to check if its solution is less than $\bar{y}_i$.

Fenchel Cuts over y variables (FC)[7]

$\hat{Y}$ is the set set of feasible depot configurations

$$\hat{Y} = \left\{ \hat{y} \in \{0,1\}^{|I|} : \sum_{i \in I} W_i \hat{y}_i \geq \sum_{j \in J} d_j \right\}.$$

Separation of $\bar{y} \in \mathrm{conv}(\hat{Y})$

We try to find $\alpha_i \in \mathbb{R}_+^{|I|}$ such that $\sum_{i \in I} \alpha_i \bar{y} < 1$, and $\sum_{i \in I} \alpha_i \hat{y} \geq 1$ for all $\hat{y} \in \hat{Y}$, by solving the LP.

$$\begin{aligned} \min \sum_{i \in I} \bar{y}_i \, \alpha_i & \\ \sum_{i \in I} \hat{y}_i \, \alpha_i \geq 1, & \quad \forall \, \hat{y} \in \hat{Y}, \\ \alpha_i \geq 0, & \quad \forall \, i \in I. \end{aligned}$$

[7] Maurizio Boccia, Antonio Sforza, Claudio Sterle, and Igor Vasilyev (2008). "A Cut and Branch Approach for the Capacitated p-Median Problem Based on Fenchel Cutting Planes". In: *Journal of Mathematical Modelling and Algorithms* 7.1, pp. 43–58.

Chvátal-Gomory Rank-1 Cuts [Jepsen et al. 2008]

Each cut is obtained by a Chvátal-Gomory rounding of a set $C \subseteq J$ of set packing constraints using a vector of multipliers ρ ($0 < \rho_j < 1, j \in C$):

$$\sum_{i \in I} \sum_{r \in R_i} \left\lfloor \sum_{j \in C} \rho_j \sum_{e \in \delta(j)} \frac{1}{2} a_e^r \right\rfloor \lambda_r^i \leq \left\lfloor \sum_{j \in C} \rho_j \right\rfloor$$

All non-dominated vectors ρ of multipliers for $|C| \leq 5$ are given in [Pecin et al. 2017].

Non-robust in the terminology of [Pessoa et al. 2008]

Separation

Enumeration for $|C| \leq 3$ and a local search heuristic for each non-dominated vector of multipliers for $|C| = \{4, 5\}$.

Other components of the BCP (through VRPSolver)

- Bucket graph-based labelling algorithm for the RCSP pricing [Righini and Salani 2006] [Sadykov et al. 2021]
- Partially elementary path (*ng*-path) relaxation [Baldacci et al. 2011]
- Automatic dual price smoothing stabilization [Wentges 1997] [Pessoa et al. 2018]
- Reduced cost fixing of (bucket) arcs in the pricing problem [Ibaraki and Nakamura 1994] [Irnich et al. 2010] [Sadykov et al. 2021]
- Enumeration of elementary routes [Baldacci et al. 2008]
- Multi-phase strong branching [Pecin et al. 2017]
  - On number of open depots in a subset of size at most 4 (largest priority)
  - On number of vehicles starting in a depot
  - On the total number of vehicles
  - On number of clients served from a depot
  - On assignment of clients to depots
  - On edges of the graph

Computational results: impact of cuts

"Classic" CLRP test instances by [Prins et al. 2006][8] with 5-10 depot locations and 50-200 clients. Time limit is 12 hours.

BCP_0 — "pure" VRPSolver (without problem-specific cuts)

	Root			Geomean	
Variant	Gap	Time (s)	Nodes	Time (s)	Solved
BCP_0	4.46%	57.9	19.2	758.7	24/26
$\text{BCP}_{\text{all}-\text{GUB}}$	3.08%	99.0	9.0	481.0	25/26
$\text{BCP}_{\text{all}-\text{DCC}}$	0.85%	101.0	9.1	504.6	24/26
$\text{BCP}_{\text{all}-\text{FC}}$	0.67%	111.4	4.4	283.9	25/26
$\text{BCP}_{\text{all}-\text{RLKC}}$	0.52%	114.7	4.1	264.0	25/26
$\text{BCP}_{\text{all}-\text{COV}}$	0.49%	114.4	4.6	273.4	25/26
BCP_{all}	0.48%	115.0	4.1	265.5	25/26

[8] Christian Prins, Caroline Prodhon, and Roberto Wolfler Calvo (2006). "Solving the capacitated location-routing problem by a GRASP complemented by a learning process and a path relinking". In: *4OR* 4.3, pp. 221–238.

Computational results: modified instances

$\rho = \frac{Q}{\sum_{i \in I} W_i / |I|}$ — "vehicle capacity / depot capacity" ratio

Variant	ρ	Root Gap	Root Time (s)	Nodes	Geomean Time (s)	Solved
BCP_0	0.3	2.14%	32.1	18.6	344.7	19/20
$BCP_{all-DCC}$	0.3	0.74%	41.0	12.7	260.9	19/20
$BCP_{all-RLKC}$	0.3	0.51%	48.2	6.1	174.1	19/20
BCP_{all}	0.3	0.46%	49.3	5.8	162.9	19/20
BCP_0	0.5	3.33%	42.9	76.7	2513.6	13/17
$BCP_{all-DCC}$	0.5	2.09%	108.4	35.0	1979.7	13/17
$BCP_{all-RLKC}$	0.5	1.73%	69.8	24.3	1059.3	14/17
BCP_{all}	0.5	1.26%	120.3	13.2	813.6	15/17
BCP_0	0.7	5.94%	51.6	255.3	10511.0	6/17
$BCP_{all-DCC}$	0.7	2.49%	247.7	58.1	4531.6	12/17
$BCP_{all-RLKC}$	0.7	3.91%	83.0	89.4	5438.7	10/17
BCP_{all}	0.7	1.53%	284.6	18.9	1734.7	14/17

Cut generation statistics

of generated cuts (# of active cuts at the end of the root)

Cut family	Original instances	Modified instances $\rho = 0.3$	Modified instances $\rho = 0.7$
RCC	492.5 (11.6)	349.5 (7.2)	254.9 (1.9)
lm-R1C	7044.0 (215.4)	22405.5 (153.0)	35610.9 (236.2)
COV	30.8 (0.2)	28.1 (0.1)	30.7 (0.1)
FC	4.0 (0.7)	3.9 (0.6)	4.4 (0.5)
GUB	338.5 (78.4)	282.4 (30.0)	203.2 (12.9)
DCC	488.4 (9.9)	1135.1 (10.1)	1636.1 (11.4)
RLKC (total)	53.4 (1.2)	325.1 (3.9)	12785.2 (29.6)
RLKCround	26.8 (0.7)	296.1 (3.7)	109.9 (0.8)
RLKC1/2	0.7 (0.0)	0.5 (0.0)	1401.8 (7.7)
RLKC1/3	0.7 (0.0)	0.8 (0.0)	675.7 (1.6)
RLKC1/4	1.2 (0.0)	1.4 (0.1)	418.2 (0.9)
RLKC1/5	1.8 (0.0)	1.5 (0.0)	1738.4 (3.8)
RLKC1/6	2.6 (0.0)	3.2 (0.1)	843.4 (1.7)
RLKC1/8	5.7 (0.1)	5.0 (0.1)	2611.1 (4.9)
RLKC1/10	13.8 (0.3)	16.6 (0.1)	4986.7 (8.2)

Computational results: comparison with the literature

Time limit is 30 hours

Instances	BCP_{all} Solved	BCP_{all} Time	[Contardo et al. 2014] Solved	[Contardo et al. 2014] Time
PPW06	24/26	518	16/26	836
TB99	9/9	945	6/9	5589

Instances by [Schneider and Löffler 2019][9]

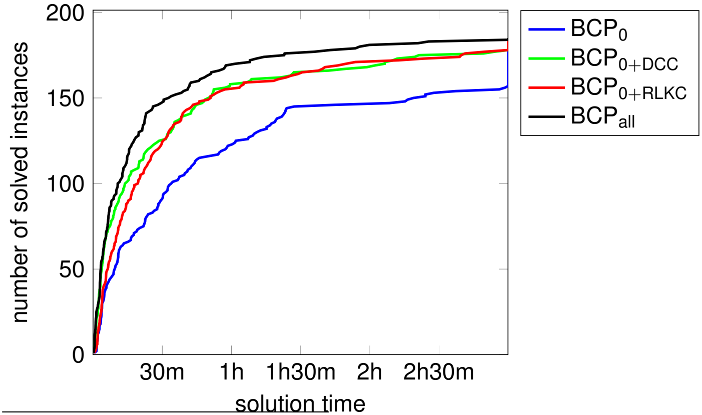
$\|I\|$	$\|J\|$	Solved	Improved BKS	Improvement
5	100	14/14	7/14	0.05%
10	100	14/14	5/14	0.11%
10	200	11/14	13/14	0.08%
15	200	15/20	18/20	0.12%
15	300	6/20	11/20	0.29%
20	300	4/20	8/20	0.91%

[9] Michael Schneider and Maximilian Löffler (2019). "Large Composite Neighborhoods for the Capacitated Location-Routing Problem". In: *Transportation Science* 53.1, pp. 301–318.

VRP-CMD instances

Instances from [Ben Mohamed et al. 2022][10], occur when solving the 2-echelon stochastic multi-period CLRP.

50 customers, 3-5 already opened depots

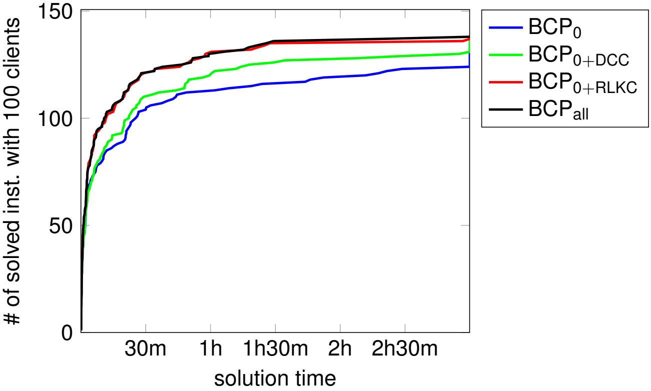


[10] Imen Ben Mohamed, Walid Klibi, Ruslan Sadykov, Halil Şen, and François Vanderbeck (2022). "The two-echelon stochastic multi-period capacitated location-routing problem". In: *European Journal of Operational Research*.

VRP with Time Windows and Shifts[11]

Instances with 25-100 customers and 3 shifts

Solved 421 from 504 instances in 30 min. ([Dabia et al. 2019] solved 280)



[11] Said Dabia, Stefan Ropke, and Tom van Woensel (2019). "Cover Inequalities for a Vehicle Routing Problem with Time Windows and Shifts". In: *Transportation Science* 53.5, pp. 1354–1371.

Conclusions

- A new family of non-robust strong knapsack cuts for the problems
- Exploited monotonicity and superadditivity properties of cuts to limit their impact on the pricing time
- These cuts make the BCP algorithm more robust (more harder instances can be solved)
- First exact algorithm for the CLRP which can scale to instances with many depot locations
- Good results for different problems with the nested knapsack structure.
- The paper has been accepted (subject to a minor revision) to the OR journal.

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